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A NOTE ON BEST AND BEST SIMULTANEOUS APPROXIMATIONS

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In this note a relationship between best approximation (A) and best simultaneous approximation in inner product spaces by elements of linear subspaces has been established. A sufficient condition for the existence of best approximation (B) in normed linear spaces by elements of linear subspaces is provided. A uniqueness theorem on best simultaneous approximation in metric linear spaces is proved.

1. INTRODUCTION

The problems of best approximation (A) and best simultaneous approximation have been studied extensively by Singer (1970), Diaz and McLaughlin (1972) and many others. A relationship between these two notions in inner product spaces and normed linear spaces is established here. A sufficient condition for the existence of best approximations (B) of every element of a general normed linear space in an arbitrary subspace is also provided. A uniqueness theorem on best simultaneous approximation in metric linear spaces is proved.

2. DEFINITIONS

Let X be a normed linear space and G a subspace of X.

Definition 2.1 — An element $g_0 \in G$ is said to be a best approximation (A) of the element $x \in X$ if and only if

$$\| x - g_0 \| \leqslant \| x - g \| \quad (g \in G).$$

Definition 2.2 — An element $g_0 \in G$ is said to be a best approximation (B) of the element $x \in X$ if and only if

$$\| g_0 - g \| \leqslant \| x - g \| \quad (g \in G).$$

Definition 2.3 — An element $g_0 \in G$ is said to be a best simultaneous approximation of the pair x_1, x_2 in X if and only if

$$\max (\| x_1 - g_0 \|, \| x_2 - g_0 \|) \leqslant \max (\| x_1 - g \|, \| x_2 - g \|) \quad (g \in G).$$

In an inner product space,

$$x \perp y \Leftrightarrow (x, y) = 0$$

whereas in a normed linear space,

$$x \perp y \Leftrightarrow \| x + ty \| \geqslant \| x \|$$

for every scalar t.

There are other definitions for the notion of orthogonality in a normed linear space, which are not used here (see James 1945).

Definition 2.4 — Let G be a subspace of an inner product space X (or a normed linear space). Then $G^{\perp}$ is defined as

$$G^{\perp} = \{x \in X : x \perp g \quad \text{for all} \quad g \in G\}.$$

3. Relationship Between Best Approximation (A) and Best Simultaneous Approximation

Let Y be a normed linear space and M a subspace of Y. Diaz and McLaughlin (1972) have proved that the element $q \in M$ which is a best approximation (A) of the arithmetic mean of F and f in Y need not be a best simultaneous approximation of the pair F, f, i.e., the equality

$$\| \tfrac{1}{2}(F + f) - q \| = \inf_{p \in M} \| \tfrac{1}{2}(F + f) - p \|$$

need not imply the equality

$$\max (\| F - q \|, \| f - q \|) = \inf_{p \in M} \max (\| F - p \|, \| f - p \|).$$

In this direction the following theorems shall be proved.

Theorem 3.1 — Let X be an inner product space and G be a subspace of X. Then every pair x_1, x_2 in $G^{\perp}$ has a best simultaneous approximation in G, which is also a best approximation (A) of the arithmetic mean of x_1 and x_2.

Proof : Let $\| x_1 \| \geqslant \| x_2 \|$ (similar proof if $\| x_2 \| \geqslant \| x_1 \|$). For all $g \in G$,

$$\begin{aligned} \| x_1 \| \geqslant \| x_2 \| &\Rightarrow \| x_1 \|^2 + \| g \|^2 \geqslant \| x_2 \|^2 + \| g \|^2 \\ &\Rightarrow \| x_1 - g \|^2 \geqslant \| x_2 - g \|^2, \text{ as } x_1, x_2 \in G^{\perp} \\ &\Rightarrow \| x_1 - g \| \geqslant \| x_2 - g \| \\ &\Rightarrow \max (\| x_1 - g \|, \| x_2 - g \|) = \| x_1 - g \|. \end{aligned}$$

Since

$$\| x_1 \|^2 \leqslant \| x_1 \|^2 + \| g \|^2 \quad (g \in G)$$

it follows that

$$\| x_1 \|^2 \leqslant \| x_1 - g \|^2 \quad (g \in G)$$

and hence

$$\| x_1 \| = \inf_{g \in G} \| x_1 - g \|$$

i.e.,

$$\max (\| x_1 \|, \| x_2 \|) = \inf_{g \in G} \max (\| x_1 - g \|, \| x_2 - g \|)$$

i.e., 0 is a best simultaneous approximation of x_1 and x_2.

Since, for all $g \in G$

$$\left\| \frac{x_1 + x_2}{2} \right\|^2 \leqslant \left\| \frac{x_1 + x_2}{2} \right\|^2 + \| g \|^2$$

it is obvious that

$$\left\| \frac{x_1 + x_2}{2} \right\|^2 \leqslant \left\| \frac{x_1 + x_2}{2} - g \right\|^2.$$

Hence

$$\left\| \frac{x_1 + x_2}{2} \right\| = \inf_{g \in G} \left\| \frac{x_1 + x_2}{2} - g \right\|$$

i.e., 0 is also a best approximation of $\frac{1}{2}(x_1 + x_2)$ in G.

Theorem 3.2 — Let B be a normed linear space and M be a subspace of B. Then every pair x_1, x_2 in $M^\perp$ has a best simultaneous approximation in M, which is also a best approximation (A) of $\frac{1}{2}(x_1 + x_2)$, if x_1 and x_2 are linearly dependent.

PROOF: Let $\| x_1 \| \geqslant \| x_2 \|$ (Similar proof if $\| x_2 \| \geqslant \| x_1 \|$). Then, for every $g \in M$ and α (scalar),

$$\begin{aligned} \max (\| x_1 \|, \| x_2 \|) &= \| x_1 \| \\ &\leqslant \| x_1 + \alpha g \| \\ &\leqslant \max (\| x_1 + \alpha g \|, \| x_2 + \alpha g \|). \end{aligned}$$

Therefore

$$\max (\| x_1 \|, \| x_2 \|) = \inf_{g \in M} \max (\| x_1 - g \|, \| x_2 - g \|$$

i.e., 0 is a best simultaneous approximation of x_1 and x_2.

Since 0 is a best approximation (A) of every element $x_1 \in M^\perp$ and since whenever $x_1 \in M^\perp$, $\alpha x_1 \in M^\perp$ (α scalar), it follows that

$$\frac{x_1 + x_2}{2} = \frac{x_1 + \lambda x_1}{2} = \left(\frac{1 + \lambda}{2} \right) x_1 \quad \text{(for some } \lambda)$$

has 0 as its best approximation (A). ∎

4. Sufficient Condition for the Existence of Best Approximation (B)

In this section the following theorem shall be proved.

Theorem 4.1 — If, for every subspace V of X there exists at least one element $x \in X \setminus V$ such that x has a best approximation (B) in V, then for any subspace V of X every element in X has a best approximation (B) in V.

For the proof, we need two lemmas.

Lemma 4.1 — If $x \notin V$ has a best approximation (B) in V, then every element of the subspace $\{x, V\}$ has a best approximation (B) in V.

Proof : Clear. ∎

Lemma 4.2 — Let V, W be two subspaces of X such that $V \subset W$. If $x \notin W$ has a best approximation (B) in W and if every element of W has a best approximation (B) in V, then x has a best approximation (B) in V.

Proof : Let y_0 be a best approximation (B) in W of x

i e., $\quad \| x - y \| \geqslant \| y_0 - y \|$ for all $y \in W$.

Let z_0 be a best approximation (B) in V of y_0

i.e., $\quad \| y_0 - z \| \geqslant \| z_0 - z \|$ for all $z \in V$.

Then

$$\| z_0 - z \| \leqslant \| y_0 - z \| \leqslant \| x - z \| \quad \text{for all} \quad z \in V$$

i.e., z_0 is a best approximation (B) of x in V. ∎

Proof of Theorem 4.1 — Let Σ be the collection of all subspaces U containing V such that every element of U has a best approximation (B) in V. Then Σ is not empty. Introduce partial ordering (set inclusion) in Σ. Clearly every chain in Σ has an upper bound in Σ. Therefore by Zorn's lemma, Σ has a maximal element in Σ. Let W be a maximal element in Σ.

Now, if $W \neq X$, let $x \notin W$. Then, by assumption, x has a best approximation (B) in W. By Lemma 4.1, every element of the subspace $\{x, W\}$ has a best approximation (B) in W. Since every element of W has a best approximation (B) in V, by Lemma 4.2, every element of the subspace $\{x, W\}$ has a best approximation (B) in V, which contradicts the maximality of W. ∎

5. Uniqueness Theorem on Best Simultaneous Approximation

Theorem 5.1 — Every convex proximinal set in a strictly convex metric linear space is Chebyshev.

PROOF: Let K be a convex proximinal set in a strictly convex metric linear space (X, d) (see Ahuja *et al.* 1977). For a given pair x_1, x_2 in X, if possible, let $k_1^*, k_2^* \in K$ be such that

$$\max \{d(k_1^*, x_1), d(k_1^*, x_2)\} = \max \{d(k_2^*, x_1), d(k_2^*, x_2)\} = r$$

where

$$r = \inf \{\max (d(k, x_1), d(k, x_2)) : k \in K\}.$$

Then

$$d(k_1^*, x_1) \leqslant r, \; d(k_2^*, x_1) \leqslant r$$

and

$$d(k_1^*, x_2) \leqslant r, \; d(k_2^*, x_2) \leqslant r.$$

X being strictly convex,

$$d\left(\frac{k_1^* + k_2^*}{2}, x_1\right) < r$$

and

$$d\left(\frac{k_1^* + k_2^*}{2}, x_2\right) < r, \quad \text{unless} \quad k_1^* = k_2^*,$$

which contradicts the definition of r, since $\frac{1}{2}(k_1^* + k_2^*) \in K$. Hence $k_1^* = k_2^*$. ∎

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REFERENCES

Ahuja, G. C., Narang, T. D., and Trehan, S. (1977). Best approximation on convex sets in metric linear spaces. *Math. Nachr.*, **78**, 125–30.

Diaz, J. B., and McLaughlin, H. W. (1972). On simultaneous Chebyshev approximation and Chebyshev approximation with an additive weight. *J. approx. Theory*, **6**, 68–71.

James, R. C. (1945). Orthogonality in normed linear spaces. *Duke Math. J.*, **12**, 291–302.

Singer, I. (1970). Best approximation in normed linear spaces by elements of linear subspaces. Springer-Verlag, New York.