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ON A STUDY OF INTEGRAL TRANSFORMS

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An attempt is made in this paper to unify and extend a large number of theorems on integral transforms, given from time to time by several authors. In all four theorems are established. The first and third theorems express relationships between images and originals of related functions in the transforms concerned while the second and fourth reveal interconnections between images of related functions in the respective transforms.

§1. A fairly large number of theorems have been added in the field of integral transforms. These can be broadly classified under two categories. The first expresses relationships between images and originals of related functions in the transforms concerned while the second reveals interconnections between images of related functions in the respective transforms. These theorems, mostly refer to integral transforms involving one variable. An attempt has been made in this paper to unify and extend these results by establishing two general theorems (involving integral transforms of two variables), embracing each of the two types mentioned above. The analogues of these theorems for integral transforms involving one variable have been given as Theorem III and IV. A number of further theorems which are new follow as special cases of our main theorems. Again, our theorems also unify and extend a large number of known theorems obtained from time to time by several authors. We give in this paper references to twenty seven such theorems.

Integral Transforms

In line with the definition of linear integral transform, the double integral transform $T\{f(x, y); p, q\}$ of a function $f(x, y)$ belonging to a prescribed class of functions of two variables is defined and represented as follows:

$$T\{f(x, y); p, q\} = \int_0^\infty \int_0^\infty k(x, y, p, q)\, f(x, y)\, dx\, dy \qquad ...(1.1)$$

where $f(x, y)$, and the parameters p, q are always so chosen that the above integral is absolutely convergent. For a specific transform, $k(x, y, p, q)$ is a definite function of x, y, p and q and is known as the kernel of the transform; $T\{f(x, y); p, q\}$ is called the image of $f(x, y)$ in the said transform, and $f(x, y)$ the original of

$T\{f(x, y); p, q\}$.

The integral transform given by (1.1) is denoted symbolically as

$$\phi(p, q) = T\{f(x, y); p, q\}.$$

For an integral transform of the type given by (1.1), it is easy to verify that if

$$\phi_1(p, q) = T\{f_1(x, y); p, q)\} \quad \text{and} \quad \phi_2(p, q) = T\{f_2(x, y); p, q\}$$

then, under certain appropriate conditions of convergence of the integrals involved, the following formula analogous to Parseval Goldstein type of formula for the Laplace transform in one and more variables holds:

$$\int_0^\infty \int_0^\infty f_2(x, y)\, \phi_1(x, y)\, dx\, dy = \int_0^\infty \int_0^\infty f_1(x, y)\, \phi_2(x, y)\, dx\, dy. \qquad \text{...(1.2)}$$

The above formula will be referred to as generalized Parseval Goldstein formula and will be required in the sequel.

§2. In this section, we establish a theorem which is sufficiently general in nature. It exhibits a close relationship between images and originals of related functions in the integral transforms T_1 and T_2 defined below:

$$T_1\{f(x, y); p, q\} = \int_0^\infty \int_0^\infty f(x, y)\, k_1(px, qy)\, dx\, dy \qquad \text{...(2.1)}$$

$$T_2\{f(x, y); p, q\} = \int_0^\infty \int_0^\infty f(x, y)\, k_2(px, qy)\, dx\, dy. \qquad \text{...(2.2)}$$

Theorem I — If

$$h_1(p, q) = T_1\{h_2(x, y)\, g(x, y); p, q\} \qquad \text{...(2.3)}$$

and

$$h_2(p^\sigma, q^\mu) = T_2\{f(x, y); p, q\} \qquad \text{...(2.4)}$$

then

$$h_1(p, q) = \sigma\mu \int_0^\infty \int_0^\infty f(x, y)\, \phi(x, y, p, q)\, dx\, dy \qquad \text{...(2.5)}$$

where

$$\phi(p, q, \alpha, \beta) = T_2\{x^{\sigma-1} y^{\mu-1} g(x^\sigma, y^\mu)\, k_1(\alpha x^\sigma, \beta y^\mu); p, q\} \qquad \text{...(2.6)}$$

the transforms T_1 and T_2 are defined by means of eqns. (2.1) and (2.2), σ, μ are non-zero real numbers of the same sign, α, β are independent of p, q and all the double integrals involved in (2.3) to (2.6) are assumed absolutely convergent.

PROOF : Applying generalized Parseval Goldstein formula given by (1.2) to the operational pairs (2.4) and (2.6), we get

$$\int_0^\infty \int_0^\infty x^{\sigma-1} y^{\mu-1} g(x^\sigma, y^\mu)\, k_1(\alpha x^\sigma, \beta y^\mu)\, h_2(x^\sigma, y^\mu)\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty \phi(x, y, \alpha, \beta)\, f(x, y)\, dx\, dy. \quad \text{...(2.7)}$$

Now, replacing α by p and β by q in (2.7), changing the variables of integration slightly on its left-hand side and interpreting the result thus obtained in terms of (2.3), we easily arrive after a little simplification at the required result (2.5).

On a suitable choice of transforms T_1, T_2 and σ in this theorem, we can easily obtain the theorems given earlier by Bose (1949b, p. 176), Goyal (1975, p. 130), Jain (1970, p. 314) and several others.

If, in the above theorem, we put $\sigma = \mu = 1$ and take T_1 and T_2 as Laplace transforms of two variables, $\phi(p, q, \alpha, \beta)$ occurring in (2.6) assumes the following form with the help of a known property of double Laplace transform:

$$\phi(p, q, \alpha, \beta) = L\{g(x, y) \exp(-\alpha x - \beta y); p, q\} = \theta(p + \alpha, q + \beta) \quad \text{...(2.8)}$$

where

$$\theta(p, q) = L\{g(x, y); p, q\}. \quad \text{...(2.9)}$$

Theorem I assumes the following form:

Theorem I(a) — If

$$h_1(p, q) = L\{g(x, y)\, h_2(x, y); p, q\} \quad \text{...(2.10)}$$

and

$$h_2(p, q) = L\{f(x, y); p, q\} \quad \text{...(2.11)}$$

then

$$h_1(p, q) = \int_0^\infty \int_0^\infty f(x, y)\, \theta(x + p, y + q)\, dx\, dy \quad \text{...(2.12)}$$

where

$$\theta(p, q) = L\{g(x, y); p, q\} \quad \text{...(2.13)}$$

Re $(p) > 0$, Re $(q) > 0$ and the various double integrals involved in (2.10) to (2.13) are absolutely convergent.

If in Theorem I(a) we take

$$h_2(x, y) = x^{\alpha-\rho} y^{\beta-\sigma} (x + A)^{-\alpha} (y + B)^{-\beta}$$

we get the following interesting corollary after a little simplification.

Corollary — If

$$\theta(p, q) = L\{g(x, y); p, q\} \qquad \ldots(2.14)$$

and

$$h(p, q) = L\{x^{\alpha-\rho}y^{\beta-\sigma}(x + A)^{-\alpha}(y + B)^{-\beta} g(x, y); p, q\} \qquad \ldots(2.15)$$

then

$$h(p, q) = [\Gamma(\rho)\,\Gamma(\sigma)]^{-1} \int_0^\infty \int_0^\infty \theta(x + p, y + q)\, x^{\rho-1}y^{\sigma-1}$$

$$\times\, {}_1F_1(\alpha; \rho; -Ax)\, {}_1F_1(\beta; \sigma; -By)\, dx\, dy \qquad \ldots(2.16)$$

provided that integrals represented by eqns. (2.14) to (2.16) are absolutely convergent, Re $(p) > 0$, Re $(q) > 0$, Re $(\rho) > 0$ and Re $(\sigma) > 0$.

§3. Now we establish the theorem which reveals interconnections between images of related functions in the transforms T_1 and T_2 defined by eqns. (2.1) and (2.2).

Theorem II — If

$$h_1(p, q) = T_1\{x^{(\rho/\sigma)-1}\, y^{(\delta/\mu)-1}\, f(x, y); p, q\} \qquad \ldots(3.1)$$

and

$$h_2(p, q) = T_2\{f(x^{-\sigma}, y^{-\mu}); p, q\} \qquad \ldots(3.2)$$

then

$$h_1(p, q) = \sigma\mu \int_0^\infty \int_0^\infty h_2(x, y)\, \phi(x, y, p, q)\, dx\, dy \qquad \ldots(3.3)$$

where

$$p^{-\rho-1}q^{-\delta-1}k_1(\alpha p^{-\sigma}, \beta q^{-\mu}) = T_2\{\phi(x, y, \alpha, \beta); p, q\} \qquad \ldots(3.4)$$

σ, μ are non-zero real numbers of the same sign, α, β are independent of p, q and the integrals involved in eqns. (3.1) to (3.4) are absolutely convergent.

PROOF : Applying generalized Parseval Goldstein formula given by (1.2) to the operational pairs (3.2) and (3.4), we get

$$\int_0^\infty \int_0^\infty x^{-\rho-1}y^{-\delta-1}k_1(\alpha x^{-\sigma}, \beta y^{-\mu})\, f(x^{-\sigma}, y^{-\mu})\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty h_2(x, y)\, \phi(x, y, \alpha, \beta)\, dx\, dy. \qquad \ldots(3.5)$$

Now, replacing α by p, β by q in (3.5), changing the variables of integration on its left-hand side and interpreting the result thus obtained in terms of (3.1), we easily get the required result (3.3).

If in Theorem II, we take both T_1 and T_2 as Laplace transforms of two variables, we get the following corollary after a little simplification:

Corollary — If

$$h_1(p, q) = L\{x^{(\rho/\sigma)-1} y^{(\delta/\mu)-1} f(x, y); p, q\} \quad \text{...(3.6)}$$

and

$$h_2(p, q) = L\{f(x^{-\sigma}, y^{-\mu}); p, q\} \quad \text{...(3.7)}$$

then

$$h_1(p, q) = \sigma\mu \int_0^\infty \int_0^\infty x^\rho y^\delta h_2(x, y)\, J_\sigma^\rho(px^\sigma)\, J_\mu^\delta(qy^\mu)\, dx\, dy \quad \text{...(3.8)}$$

where $\text{Re}(p) > 0$, $\text{Re}(q) > 0$, $\sigma > 0$, $\mu > 0$, $\text{Re}(\rho + 1) > 0$, $\text{Re}(\delta + 1) > 0$ and the integrals represented by eqns. (3.6) to (3.8) are absolutely convergent. In (3.8) $J_v^\mu(x)$ denotes Wright's generalized Bessel function.

§4. *Integral Transforms Involving one Variable* — If we take all the integral transforms occurring in Theorems I and II to be analogous integral transforms involving one variable, we easily arrive, on proceeding along the lines followed earlier, at the following results concerning integral transforms of one variable. We, however, have not given their separate proofs.

Theorem III — If

$$h_1(p) = T_1\{h_2(x)\, g(x); p\} = \int_0^\infty k_1(px)\, h_2(x)\, g(x)\, dx \quad \text{...(4.1)}$$

and

$$h_2(p^\sigma) = T_2\{f(x); p\} = \int_0^\infty k_2(px)\, f(x)\, dx \quad \text{...(4.2)}$$

then

$$h_1(p) = \sigma \int_0^\infty f(x)\, \phi(x, p)\, dx \quad \text{...(4.3)}$$

where

$$\phi(p, \alpha) = T_2\{x^{\sigma-1} g(x^\sigma)\, k_1(\alpha x^\sigma); p\}$$

$$= \int_0^\infty k_2(px)\, x^{\sigma-1} g(x^\sigma)\, k_1(\alpha x^\sigma);\, dx \quad \text{...(4.4)}$$

provided that the integrals involved in eqns. (4.1) to (4.4) are absolutely convergent, α is independent of p, and σ is a non-zero real number.

Special Cases of Theorem III

Theorem III given above is quite general in nature and is of great interest in itself. It generalizes a large number of interesting theorems given from time to time by several authors lying scattered in the literature. Thus, on making suitable choice of transforms T_1, T_2, function $g(x)$ and the quantity σ in Theorem III, we easily get from it, the theorems obtained earlier by Bose (1949a, pp. 59, 64), Kaushik (1946, p. 7), Jaiswal (1952, p. 85), Kesarwani (1957, p. 298), Saxena (1960, p. 404; 1961, p. 292), Gupta (1976, p. 3; 1964, p. 165), Srivastava and Vyas (1969, p. 140), Kalla and de Battig (1972, p. 159), Sharma (1958, p. 114), Mittal (1971, p. 33), Koul (1972, p. 14), Arya (1959, p. 40), Varma (1959, p. 204), Srivastava (1972, p. 61), Saksena (1953) and many others.

If we put $\sigma = 1$ in Theorem III we get a theorem obtained earlier by Agarwal (1970, p. 538).

Theorem IV — If

$$h_1(p) = T_1 \{x^{(\rho/\sigma)-1} f(x); p\} = \int_0^\infty k_1(px)\, x^{(\rho/\sigma)-1} f(x)\, dx \qquad \ldots(4.5)$$

and

$$h_2(p) = T_2 \{f(x^{-\sigma}); p\} = \int_0^\infty k_2(px)\, f(x^{-\sigma})\, dx \qquad \ldots(4.6)$$

then

$$h_1(p) = \sigma \int_0^\infty h_2(x)\, \phi(x, p)\, dx \qquad \ldots(4.7)$$

where

$$p^{-\rho-1} k_1(\alpha p^{-\sigma}) = T_2 \{\phi(x, \alpha) : p\} \qquad \ldots(4.8)$$

σ is a non-zero real number, α is independent of p and the integrals involved in eqns. (4.5) to (4.8) are absolutely convergent.

Theorem IV given above is also sufficiently general in nature and generalizes a number of scattered theorems in the literature on making suitable choice of transforms T_1, T_2 and σ in it. Thus the theorems obtained earlier by Bose (1950, p. 18), Tiwari (1943, p. 33), Rathie (1955, p. 385), Gupta (1948, p. 140), Saxena and Gupta (1964, p. 712), Saxena (1959, p. 166) and several others follow as special cases of Theorem IV.

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REFERENCES

Agarwal, R. P. (1970). On certain transformation formulae and Meijer's G-function of two variables. *Indian J. pure appl. Math.*, **1**, 537–51.

Arya, S. C. (1959). Some theorems connected with a generalized Stieltjes transform. *Bull. Calcutta math. Soc.*, **51**, 39–47.

Bose, S. K. (1949a). A study of the generalized Laplace integral II. *Bull. Calcutta math. Soc.*, **41**, 59–67.

———— (1949b). On Laplace transform of two variables. *Bull. Calcutta math. Soc.*, **41**, 173–78.

———— (1950). A note on Whittaker transform. *Ganita*, **1**, 16–22.

Goyal, S. P. (1975). Study of a generalized integral operator$^{(1)}$. *Portugal Math.*, **34**, 127–47.

Gupta, H. C. (1948). On operational Calculus. *Proc. natn. Inst. Sci. India*, **A3**, 131–56.

Gupta, K. C. (1964). A theorem concerning Meijer and Varma transforms. *Proc. Natn. Acad. Sci., India*, **A34**, 163–68.

———— (1976). A theorem on integral transforms. *Riv. Mat. Univ. Parma*, **2**, 1–14.

Jain, N. C. (1970). A relation between Laplace and Stieltjes transform of two variables. *Ann. Polon. Math.*, **22**, 313–15.

Jaiswal, J. P. (1952). Two properties of Meijer transform. *Ganita*, **3**, 85–90.

Kalla, S. L., and De Battig, N. E. F. (1972). Algunos theoremas sobre la transformada cuyo nucleo es la funcion H. *Univ. Nac. Tucuman Rev.*, A **22**, 157–63.

Kaushik, S. P. (1946). A theorem on the generalized Laplace's transform. *Proc. Benaras Math. Soc.*, **8**, 7–10.

Kesarwani, R N. (1957). Some properties of generalized Laplace transform I. *Riv. Mat. Univ. Parma*, **8**, 283–306.

Koul, C. L. (1972). A theorem relating generalized Hankel transform and a Whittaker transform. *Math. Student*, **40**, 13–17.

Mittal, P. K. (1971). Certain properties of Meijer's G-function transform involving the H-function. *Vijnana Parishad Anusandhan Patrika*, **14**, 29–34.

Rathie, C. B. (1955). Some properties of generalized Laplace transform. *Proc. natn. Inst. Sci. India*, **A21**, 382–93.

Saksena, K. M. (1953). Generalizations of Stieltjes transform. *Bull. Calcutta math. Soc.*, **45**, 101–7.

Saxena, R. K. (1959). A theorem on Meijer transform. *Proc. natn. Inst. Sci. India*, **A25**, 166–70.

———— (1960). Some theorems on generalized Laplace transform—I. *Proc. natn. Inst. Sci. India*, **A26**, 400–13.

———— (1961). Some theorems on generalized Laplace transform—II. *Riv. Mat. Univ. Parma*, **2**, 287–300.

Saxena, R. K., and Gupta, K. C. (1964). Certain properties of generalized Stieltjes transform involving Meijer's G-function. *Proc. natn. Inst. Sci., India*, **A24**, 707–14.

Sharma, K. C. (1958). A theorem on Meijer transform and infinite integrals involving G-function and Bessel functions. *Prac. natn. Inst. Sci. India*, **A24**, 113–20.

Srivastava, H. M., and Vyas, O. D. (1969). A theorem relating generalized Hankel and Whittaker transforms. *Nederl. Akad. Wetensch. Proc.*, A **72**, 140–44.

Srivastava, S. K. (1972). Theorems on a generalized Laplace transform. *Acta Mexicana Ci. Tech.*, **6**, 59–63.

Tiwari, N. D. (1943). A theorem in operational calculus. *Proc. Banaras Math. Soc.*, **5**, 33–36.

Varma, C. B. L. (1959). On some properties of K-transform involving Meijer's G-function. *Proc. natn. Acad. Sci. India*, **A28**, 200–207.