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# ON CERTAIN DISCRETE INEQUALITIES OF THE WENDROFF TYPE

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The aim of this paper is to establish some new discrete inequalities involving three independent variables which can be used in the study of various problems in the theory of discrete versions of the hyperbolic partial differential and integral equations involving three independent variables.

## 1. INTRODUCTION

One reason for much of the successful mathematical development of the classical theories of finite difference equations and numerical analysis is the availability of some kinds of discrete inequalities (Diaz 1958; Henrici 1962; Hull and Luxemburg 1960; Jones 1964; Pachpatte 1973a, b, 1977 a–c). There have been in the past few years an increasing number of applications of the discrete inequalities to a variety of mathematical problems of physical interest. While progress has been made in recent years in the theory of finite difference equations and numerical analysis, there is an unfortunate shortage of discrete inequalities which are readily applicable to the discrete versions of the partial differential and integral equations. Our objective here is to establish some new discrete inequalities involving three independent variables which can be used in the study of discrete versions of partial differential and integral equations involving three independent variables.

## 2. MAIN RESULTS

Before giving the main results in this section, we first recollect a few of the notions and definitions from Pachpatte (1973a, b, 1977a–c). Let $N$ be the set of points $n_0 + k$ $(k = 0, 1, 2, \ldots)$, where $n_0 \geqslant 0$ is a given integer. The expression $u(n_0) + \sum_{s=n_0}^{n-1} b(s)$ represents a solution of the linear difference equation $\Delta u(n) = b(n)$ for all $n \in N$, where $\Delta$ is the operator defined by $\Delta u(n) = u(n+1) - u(n)$. It is supposed that $\sum_{s=n_0}^{n_0-1} b(s) = 0$. The expression $u(n_0) \prod_{s=n_0}^{n-1} c(s)$ represents a solution of the linear difference equation $u(n+1) = c(n)\, u(n)$ for all $n \in N$. It is supposed that $\prod_{s=n_0}^{n_0-1} c(s) = 1$. In our subsequent discussion we also use the following notions of the operators

$$\Delta_x [u(x, y, z)] = \Delta u_x(x, y, z) = u(x+1, y, z) - u(x, y, z),$$
$$\Delta_y [u(x, y, z)] = \Delta u_y(x, y, z) = u(x, y+1, z) - u(x, y, z),$$
$$\Delta_z [u(x, y, z)] = \Delta u_z(x, y, z) = u(x, y, z+1) - u(x, y, z),$$
$$\Delta_y [\Delta u_x(x, y, z)] = \overset{2}{\Delta} u_{xy}(x, y, z) = \Delta u_x(x, y+1, z) - \Delta u_x(x, y, z)$$

and so on. We often use the letters $x$, $y$ and $z$ to denote the three independent variables which are the members of $N$.

A useful three independent variable discrete inequality is embodied in the following theorem.

*Theorem* 1 — Let $u(x, y, z)$ and $b(x, y, z)$ be real-valued nonnegative functions defined for $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, and let $a(x, y, z)$ be positive nondecreasing in all the three variables, and defined for $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, for which the inequality

$$u(x, y, z) \leqslant a(x, y, z) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\, u(s, t, r) \qquad \dots(1)$$

holds for $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$. Then

$$u(x, y, z) \leqslant a(x, y, z) \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\right] \qquad \dots(2)$$

for all $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

PROOF: Since $a(x, y, z)$ is positive, nondecreasing, we observe from (1) that

$$\frac{u(x, y, z)}{a(x, y, z)} \leqslant 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \frac{u(s, t, r)}{a(x, y, z)}$$

$$\leqslant 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \frac{u(s, t, r)}{a(s, t, r)}. \qquad \dots(3)$$

Define a function $m(x, y, z)$ by

$$m(x, y, z) = 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \frac{u(s, t, r)}{a(s, t, r)},$$

$$m(0, y, z) = m(x, 0, z) = m(x, y, 0) = 1$$

then

$$\Delta m_x(x, y, z) = \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(x, t, r) \frac{u(x, t, r)}{a(x, t, r)} \qquad \dots(4)$$

and from (4) we have

$$\triangle m_x(x, y+1, z) - \triangle m_x(x, y, z) = \sum_{r=0}^{z-1} b(x, y, r) \frac{u(x, y, r)}{a(x, y, r)} \qquad \ldots(5)$$

$$\triangle m_x(x, y+1, z+1) - \triangle m_x(x, y, z+1) = \sum_{r=0}^{z} b(x, y, r) \frac{u(x, y, r)}{a(x, y, r)}. \qquad \ldots(6)$$

From (5) and (6) we have

$$\overset{2}{\triangle} m_{xy}(x, y, z+1) - \overset{2}{\triangle} m_{xy}(x, y, z) = b(x, y, z) \frac{u(x, y, z)}{a(x, y, z)}$$

which in view of (3) implies

$$\overset{2}{\triangle} m_{xy}(x, y, z+1) - \overset{2}{\triangle} m_{xy}(x, y, z) \leqslant b(x, y, z)\, m(x, y, z). \qquad \ldots(7)$$

From the definition of $m(x, y, z)$ we observe that $m(x, y, z) \leqslant m(x, y, z+1)$, for $x \geqslant 0, y \geqslant 0, z \geqslant 0$. Using this fact in (7) we have

$$\overset{2}{\triangle} m_{xy}(x, y, z+1) - \overset{2}{\triangle} m_{xy}(x, y, z) \leqslant b(x, y, z)\, m(x, y, z+1)$$

i.e.

$$\frac{\overset{2}{\triangle} m_{xy}(x, y, z+1)}{m(x, y, z+1)} - \frac{\overset{2}{\triangle} m_{xy}(x, y, z)}{m(x, y, z+1)} \leqslant b(x, y, z). \qquad \ldots(8)$$

From (8) we observe that

$$\frac{\overset{2}{\triangle} m_{xy}(x, y, z+1)}{m(x, y, z+1)} - \frac{\overset{2}{\triangle} m_{xy}(x, y, z)}{m(x, y, z)} \leqslant b(x, y, z). \qquad \ldots(9)$$

Now keeping $x, y$ fixed in (9), setting $z = r$ and substituting $r = 0, 1, 2, \ldots, z-1$ we obtain the estimate for $\dfrac{\overset{2}{\triangle} m_{xy}(x, y, z)}{m(x, y, z)}$ such that

$$\frac{\overset{2}{\triangle} m_{xy}(x, y, z)}{m(x, y, z)} \leqslant \sum_{r=0}^{z-1} b(x, y, r). \qquad \ldots(10)$$

From (10) and in view of the fact that $m(x, y, z) \leqslant m(x, y+1, z)$ we observe that

$$\frac{\triangle m_x(x, y+1, z)}{m(x, y+1, z)} - \frac{\triangle m_x(x, y, z)}{m(x, y, z)} \leqslant \sum_{r=0}^{z-1} b(x, y, r). \qquad \ldots(11)$$

Keeping $x$, $z$ fixed in (11), setting $y = t$ and substituting $t = 0, 1, 2, \ldots, y-1$ we obtain the estimate for $\dfrac{\Delta\, m_x(x, y, z)}{m(x, y, z)}$ such that

$$\frac{\Delta\, m_x(x, y, z)}{m(x, y, z)} \leqslant \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(x, t, r). \qquad \ldots(12)$$

From (12) we have

$$m(x+1, y, z) \leqslant m(x, y, z) \left[ 1 + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(x, t, r) \right]. \qquad \ldots(13)$$

Again keeping $y$, $z$ fixed in (13), setting $x = s$ and substituting $s = 0, 1, 2, \ldots, x-1$ we obtain the estimate for $m(x, y, z)$ such that

$$m(x, y, z) \leqslant \prod_{s=0}^{x-1} \left[ 1 + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \right].$$

Substituting this bound on $m(x, y, z)$ in (3) we obtain the desired bound in (2).

We now apply Theorem 1 to establish the following useful discrete inequality in three independent variables.

*Theorem* 2 — Let $u(x, y, z)$, $b(x, y, z)$, and $c(x, y, z)$ be real-valued nonnegative functions defined for $x \geqslant 0, y \geqslant 0, z \geqslant 0$; and let $W(u)$ be continuous, positive strictly increasing function on $I = [u_0, \infty)$, $u_0 > 0$, and suppose further that the inequality

$$u(x, y, z) \leqslant M + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\, u(s, t, r)$$
$$+ \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} c(s, t, r)\, W(u(s, t, r)) \qquad \ldots(14)$$

is satisfied for $x \geqslant 0, y \geqslant 0, z \geqslant 0$, where $M > 0$ is a constant. Then for $0 \leqslant x \leqslant x_1, 0 \leqslant y \leqslant y_1, 0 \leqslant z \leqslant z_1$,

$$u(x, y, z) \leqslant \Omega^{-1} \left[ \Omega(M) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} c(s, t, r)\, W(P(s, t, r)) \right] P(x, y, z) \qquad \ldots(15)$$

where

$$P(x, y, z) = \prod_{s=0}^{x-1} \left[ 1 + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \right] \qquad \ldots(16)$$

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{W(s)},\; r \geqslant r_0 > 0 \qquad \ldots(17)$$

$\Omega^{-1}$ is the inverse function of $\Omega$, and

$$\Omega(M) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} c(s, t, r)\, W(P(s, t, r)) \in \text{Dom}\,(\Omega^{-1})$$

for all $x, y, z$ lying in the subintervals $0 \leqslant x \leqslant x_1$, $0 \leqslant y \leqslant y_1$, $0 \leqslant z \leqslant z_1$ of $N$.

PROOF : Define

$$a(x, y, z) = M + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} c(s, t, r)\, W(u(s, t, r)),$$

$$a(0, y, z) = a(x, 0, z) = a(x, y, 0) = M \qquad \text{...(18)}$$

then (14) can restated as

$$u(x, y, z) \leqslant a(x, y, z) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\, u(s, t, r).$$

Since $a(x, y, z)$ is positive, nondecreasing, we have from Theorem 1

$$u(x, y, z) \leqslant a(x, y, z) \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\right]$$

$$= a(x, y, z)\, P(x, y, z). \qquad \text{...(19)}$$

Further

$$W(u(x, y, z)) \leqslant W(a(x, y, z))\, W(P(x, y, z))$$

since $W$ is submultiplicative. Hence,

$$\frac{c(x, y, z)\, W(u(x, y, z))}{W(a(x, y, z))} \leqslant c(x, y, z)\, W(P(x, y, z))$$

and because of (18), this reduces to

$$\frac{\overset{2}{\triangle}\, a_{xy}(x, y, z+1) - \overset{2}{\triangle}\, a_{xy}(x, y, z)}{W(a(x, y, z))} \leqslant c(x, y, z)\, W(P(x, y, z)). \qquad \text{...(20)}$$

From (20) we observe that

$$\frac{\overset{2}{\triangle}\, a_{xy}(x, y, z+1)}{W(a(x, y, z+1))} - \frac{\overset{2}{\triangle}\, a_{xy}(x, y, z)}{W(a(x, y, z))} \leqslant c(x, y, z)\, W(P(x, y, z)). \qquad \text{...(21)}$$

Now keeping $x, y$ fixed in (21), setting $z = r$ and substituting $r = 0, 1, 2, \ldots, z-1$ in (21) we obtain the estimate for $\dfrac{\overset{2}{\triangle}\, a_{xy}(x, y, z)}{W(a(x, y, z))}$ such that

$$\frac{\overset{2}{\triangle}\, a_{xy}(x, y, z)}{W(a(x, y, z))} \leqslant \sum_{r=0}^{z-1} c(x, y, r)\, W(P(x, y, r)). \qquad \text{...(22)}$$

From (22) we observe that

$$\frac{\triangle a_x(x, y+1, z)}{W(a(x, y+1, z))} - \frac{\triangle a_x(x, y, z)}{W(a(x, y, z))} \leqslant \sum_{r=0}^{z-1} c(x, y, r)\, W(P(x, y, r)). \quad \ldots(23)$$

Keeping $x$, $z$ fixed in (23) and setting $y = t$ and substituting $t = 0, 1, 2, \ldots, t-1$ in (23) we obtain the estimate for $\frac{\triangle a_x(x, y, z)}{W(a(x, y, z))}$ such that

$$\frac{\triangle a_x(x, y, z)}{W(a(x, y, z))} \leqslant \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} c(x, t, r)\, W(P(x, t, r)). \quad \ldots(24)$$

From (17) and (24) we have

$$\Omega(a(x+1, y, z)) - \Omega(a(x, y, z)) = \int_{a(x, y, z)}^{a(x+1, y, z)} \frac{ds}{W(s)}$$

$$\leqslant \frac{\triangle a_x(x, y, z)}{W(a(x, y, z))}$$

$$\leqslant \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} c(x, t, r)\, W(P(x, t, r)).$$

Again keeping $y$, $z$ fixed in the above inequality and setting $x = s$ and substituting $s = 0, 1, 2, \ldots, x-1$ we obtain

$$\Omega(a(x, y, z)) - \Omega(M) \leqslant \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} c(s, t, r)\, W(P(s, t, r)). \quad \ldots(25)$$

The desired bound in (15) now follows by substituting the bound on $a(x, y, z)$ from (25) in (19). The subintervals of $N$ for $x$, $y$, and $z$ are obvious.

We next establish the following three independent variable generalization of the discrete inequality recently established by Pachpatte (1973b, Theorem 1).

*Theorem* 3 — Let $u(x, y, z)$, $b(x, y, z)$, and $c(x, y, z)$ be as defined in Theorem 2; and let $a(x, y, z)$ be as defined in Theorem 1, for which the inequality

$$u(x, y, z) \leqslant a(x, y, z) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \Big[ u(s, t, r) + \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} c(k, l, n)\, u(k, l, n) \Big] \quad \ldots(26)$$

holds for $x \geqslant 0, y \geqslant 0, z \geqslant 0$. Then

$$u(x, y, z) \leqslant a(x, y, z)\left[1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\, Q(s, t, r)\right] \quad ...(27)$$

for all $x \geqslant 0, y \geqslant 0, z \geqslant 0$, where

$$Q(x, y, z) = \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} [b(s, t, r) + c(s, t, r)]\right] \quad ...(28)$$

for $x \geqslant 0, y \geqslant 0, z \geqslant 0$.

PROOF : Since $a(x, y, z)$ is positive, nondecreasing we observe from (26) that

$$\frac{u(x, y, z)}{a(x, y, z)} \leqslant 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\left[\frac{u(s, t, r)}{a(s, t, r)} + \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} c(k, l, n)\frac{u(k, l, n)}{a(k, l, n)}\right]. \quad ...(29)$$

Define a function $m(x, y, z)$ by

$$m(x, y, z) = 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\left[\frac{u(s, t, r)}{a(s, t, r)} + \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} c(k, l, n)\frac{u(k, l, n)}{a(k, l, n)}\right],$$

$$m(0, y, z) = m(x, 0, z) = m(x, y, 0) = 1.$$

Then by following the similar argument as in the proof of Theorem 1 we have

$$\overset{2}{\triangle} m_{xy}(x, y, z + 1) - \overset{2}{\triangle} m_{xy}(x, y, z) \leqslant b(x, y, z)\left[m(x, y, z) + \sum_{k=0}^{x-1} \sum_{l=0}^{y-1} \sum_{n=0}^{z-1} c(k, l, n)\, m(k, l, n)\right]. \quad ...(30)$$

If we put

$$v(x, y, z) = m(x, y, z) + \sum_{k=0}^{x-1} \sum_{l=0}^{y-1} \sum_{r=0}^{z-1} c(k, l, n)\, m(k, l, n),$$

$$v(0, y, z) = v(x, 0, z) = v(x, y, 0) = 1 \quad ...(31)$$

then again by following the similar argument as in the proof of Theorem 1 and using (30) and the fact that $m(x, y, z) \leqslant v(x, y, z)$ from (31) we have

$$\overset{2}{\triangle}\, v_{xy}(x, y, z + 1) - \overset{2}{\triangle}\, v_{xy}(x, y, z) \leqslant [b(x, y, z) + c(x, y, z)]\, v(x, y, z)$$

which by following the same technique as in the proof of Theorem 1 yields the estimate for $v(x, y, z)$ such that

$$v(x, y, z) \leqslant \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} [b(s, t, r) + c(s, t, r)]\right] = Q(x, y, z).$$

Substituting this bound on $v(x, y, z)$ in (30) and once again by following the last argument as in the proof of Theorem 1 we obtain the estimate for $m(x, y, z)$ such that

$$m(x, y, z) \leqslant 1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r)\, Q(s, t, r).$$

Substituting this bound on $m(x, y, z)$ in (29) we obtain the desired bound in (27).

We now apply Theorem 3 to establish the following more general inequality which can be used in some applications.

*Theorem* 4 — Let $u(x, y, z)$, $b(x, y, z)$, $c(x, y, z)$, and $p(x, y, z)$ be real-valued nonnegative functions defined for $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$; let $W(u)$ be the same function as defined in Theorem 2, and suppose further that the inequality

$$\begin{aligned} u(x, y, z) \leqslant M &+ \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \Big[ u(s, t, r) \\ &+ \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} c(k, l, n)\, u(k, l, n) \Big] \\ &+ \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s, t, r)\, W(u(s, t, r)) \end{aligned} \qquad \ldots(32)$$

is satisfied for $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, where $M > 0$ is a constant. Then for $0 \leqslant x \leqslant x_2$, $0 \leqslant y \leqslant y_2$, $0 \leqslant z \leqslant z_2$,

$$\begin{aligned} u(x, y, z) \leqslant \Omega^{-1} &\left[ \Omega(M) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s, t, r)\, W(R(s, t, r)) \right] \\ &\times R(x, y, z) \end{aligned} \qquad \ldots(33)$$

where $\Omega$, $\Omega^{-1}$ are as defined in Theorem 2, and

$$\begin{aligned} R(x, y, z) = 1 &+ \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} b(s, t, r) \\ &\times \prod_{k=1}^{s-1} \left[ 1 + \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} [b(k, l, n) + c(k, l, n)] \right] \end{aligned} \qquad \ldots(34)$$

and

$$\Omega(M) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s, t, r)\, W(R(s, t, r)) \in \mathrm{Dom}\,(\Omega^{-1})$$

for all $x, y, z$ lying in the subintervals $0 \leqslant x \leqslant x_2$, $0 \leqslant y \leqslant y_2$, $0 \leqslant z \leqslant z_2$ of $N$.

The proof of this theorem follows by the similar argument as in the proof of Theorem 2, by making use of Theorem 3. We omit the details.

In concluding this section we note that discrete inequalities established in Theorems 1–4 can be extended very easily to the case of $n$ independent variables. Since this translation is quite straightforward in view of the proofs of Theorems 1–4 given in this section, we leave it for the reader to fill in where needed.

## 3. Some Applications

In this section we indicate some applications of our results to obtain the bounds on the solutions of discrete versions of partial integrodifferential equations involving three independent variables. We believe that the discrete inequalities established in this paper may be used in the theory of finite difference equations involving three independent variables in essentially the same capacity as the inequalities of the Gronwall and Bihari type (Beckenbach and Bellman 1961, p. 135) are used in the theory of ordinary differential and integral equations. To illustrate the application of Theorem 4 we establish the bound on the solutions of discrete versions of partial integrodifferential equations involving three independent variables of the form

$$\overset{3}{\Delta}\, u_{xyz} = f[x, y, z, u] + F\left[x, y, z, u, \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} h(x, y, z, s, t, r, u)\right] \qquad \ldots(35)$$

with the given boundary conditions at $x = 0$, $y = 0$, $z = 0$, where all the functions are defined on their respective domains of definitions and

$$|\, f\,[x, y, z, u] \,| \leqslant p(x, y, z)\, W(\,|\, u \,|\,) \qquad \ldots(36)$$

$$|\, F[x, y, z, u, v] \,| \leqslant b(x, y, z)\, [\,|\, u \,| + |\, v \,|\,] \qquad \ldots(37)$$

$$|\, h(x, y, z, s, t, r, u) \,| \leqslant c(s, t, r)\, |\, u \,| \qquad \ldots(38)$$

for $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, where $W$, $b(x, y, z)$, $c(x, y, z)$, and $p(x, y, z)$ are as defined in Theorem 4. By using the given boundary conditions, eqn. (35) can be represented by equivalent summary difference equation

$$u(x, y, z) = g(x, y, z) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} f[s, t, r, u(s, t, r)] +$$

*(equation continued on p. 736)*

$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} F\Big[s, t, r, u(s, t, r),$$

$$\sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} h(s, t, r, k, l, n, u(k, l, n))\Big] \qquad \ldots(39)$$

where $g(x, y, z)$ depends on the given boundary conditions. If $|g(x, y, z)| \leqslant M$ (where $M > 0$ is a constant), then using (36), (37), and (38) in (39) and then applying Theorem 4, we obtain the bound on the solution $u(x, y, z)$ of (35).

In concluding this paper we note that the discrete inequalitiy established in Theorem 2 can be used to obtain the bound on the solutions of eqn. (35), when $h \equiv 0$ under the conditions (36), and (37) on the functions $f$ and $F$ involved in (35).

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