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ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES BY NÖRLUND MEANS

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In this paper a result of Chow (1954, Theorem 1) has been generalized to the absolute Nörlund summability factors of infinite series following Ahmad and Khan (1974).

§1. *Definitions and Notations* — Let $\Sigma\, a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its partial sums. Let $\{p_n\}$ be the sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + \dots + p_n, \quad (n \geqslant 0)$$

$$P_n = p_n = 0, \qquad (n = -1, -2, \dots).$$

Then, the transformation

$$\tau_n = (P_n)^{-1} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu, \qquad (P_n \neq 0)$$

defines the sequence $\{\tau_n\}$ of Nörlund means of the sequence $\{s_n\}$ (Nörlund 1919, Woronoi 1932). If $\{\tau_n\} \in BV$, that is, $\Sigma \mid \tau_n - \tau_{n-1} \mid \leqslant K^*$, then $\Sigma\, a_n$ is said to be absolutely summable (N, p_n), or simply summable $\mid N, p_n \mid$ (see Mears 1935).

The necessary and sufficient conditions for the regularity of (N, p_n)-method, are

$$p_n = o(P_n) \qquad \dots(1.1)$$

and

$$\sum_{\nu=0}^{n} \mid p_\nu \mid = O(\mid P_n \mid). \qquad \dots(1.2)$$

We observe that, from (1.2)

$$\lim_{n\to\infty} \frac{P_{n-1}}{P_n} = \lim_{n\to\infty} \left(1 - \frac{p_n}{P_n}\right) = 1. \qquad \dots(1.3)$$

In the special cases in which

$$p_n = A_n^{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)}, \quad (\alpha > -1)$$

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*K denotes throughout an absolute constant, not necessarily the same at each occurrence.

$$p_n = (n+1)^{-1}, P_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + (n+1)^{-1} \sim \log n, \quad \text{as } n \to \infty$$

the (N, p_n)-mean reduces simply to (C, α)-mean and harmonic mean respectively.

We denote the nth Cesàro mean of order $\beta(> -1)$ of the sequence $\{na_n\}$ by t_n^β, defined by

$$t_n^\beta = (A_n^\beta)^{-1} \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} \nu a_\nu.$$

For any sequence $\{u_n\}$, we write

$$\triangle^0(u_n) = u_n,\ \triangle(u_n) = \triangle^1(u_n) = u_n - u_{n+1},\ \triangle^r\triangle^s(u_n) = \triangle^{r+s}(u_n) \quad (r, s = 1, 2, \dots)$$

and

$$\triangle^k(u_n) = \sum_{\nu=0}^{\infty} A_\nu^{-k-1} u_{\nu+n}$$

provided the series on the right converges.

For $p \equiv p_n$, we write

$$\left.\begin{array}{l} N^\beta(p; n, \rho) \\ \bar{N}^\beta(p; n, \rho) \end{array}\right\} = \sum_{\nu=0}^{n} A_\nu^{-\beta-1} (P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho}) \left\{\begin{array}{l} \epsilon_{\nu+\rho}/(\nu+\rho) \\ \epsilon_{\nu+\rho}/(\nu+\rho)^2 \end{array}\right.$$

§2. The following theorem was proved by Chow (1954, Theorem 1).

Theorem A — If $0 \leqslant \alpha \leqslant \beta$ and $\{\lambda_n\}$ is a sequence of positive numbers such that $\{\lambda_n/n\}$ is non-increasing, the necessary and sufficient conditions that $\Sigma\, a_n\epsilon_n$ should be summable $| C, \alpha |$, whenever $\Sigma\, n^{-1}\lambda_n \mid t_n^\beta \mid < \infty$, are:

(i) $\epsilon_n = O(n^{\alpha-\beta}\lambda_n)$;

(ii) $\triangle^\beta(n^{-1}\epsilon_n) = O(n^{-\beta-1}\lambda_n)$.

For $0 < \alpha \leqslant \beta$, we generalize the above result for absolute Nörlund summability by replacing $| C, \alpha |$ by $| N, p_n |$, in a manner similar to that adopted by Ahmad and Khan (1974) for the generalization of another result of Chow (1954, Theorem 2).

§3. We establish the following theorem.

Theorem 1 — Let $\beta > 0$ and $\{\lambda_n\}$ be the sequence of numbers such that $\lambda_n > 0$ and $\{\lambda_n/n\}$ is non-increasing, and let the sequence $\{p_n\}$ be nonnegative and non-increasing such that $p_0 > 0$ and $\{P_n/n^\beta\}$ is bounded. Then the necessary and sufficient conditions such that $\Sigma a_n \epsilon_n$ should be summable $| N, p_n |$ whenever $\Sigma n^{-1} \lambda_n | t_n^{\beta} | < \infty$, are:

(i) $\epsilon_n = O(n^{-\beta} P_n \lambda_n)$;

(ii) $\triangle^{\beta}(n^{-1}\epsilon_n) = O(n^{-\beta-1}\lambda_n)$.

Remark : Since $\{p_n\}$ is nonnegative and non-increasing such that $p_0 > 0$, the condition of boundedness of the sequence $\{P_n/n^{\beta}\}$ is automatically satisfied when $\beta \geqslant 1$, therefore in the theorem this condition should be treated as void for $\beta \geqslant 1$.

§4. For the proof of the theorem the following lemmas are needed.

Lemma 1 (Chow 1954, Lemma 4) — Let $\{S_{n,\mu}\}$ be a double sequence. If

$$U_n = \sum_{\mu=1}^{\infty} S_{n,\mu} u_{\mu}, \quad (n = 1, 2, 3, \ldots)$$

a necessary and sufficient condition that the series $\Sigma | U_n |$ should be convergent whenever the series $\Sigma | u_n |$ is convergent, is that

$$\sum_{n=1}^{\infty} | S_{n,\mu} | \leqslant C$$

where C is a constant independent of μ.

Lemma 2 (Chow 1954, Lemma 1) — If $\sigma > -1$ and $\sigma - \delta > 0$, then

$$\sum_{n=\mu}^{\infty} \frac{A_{n-\mu}^{\delta}}{n A_n^{\sigma}} = \sum_{n=0}^{\infty} \frac{A_n^{\delta}}{(n+\mu) A_{n+\mu}^{\sigma}} = \frac{1}{\mu A_{\mu}^{\sigma-\delta-1}}.$$

Lemma 3 — If $p_0 > 0$, and $\{p_n\}$ is nonnegative and non-increasing, then for $\nu \geqslant 1$

(a) $$\sum_{n=\nu}^{\infty} \frac{1}{P_n P_{n-1}} (P_n p_{n-\nu} - P_{n-\nu} p_n) \leqslant K;$$

(b) $$\sum_{n=\nu}^{\infty} \frac{| \Delta_n p_{n-\nu-1} |}{P_n P_{n-1}} \leqslant \frac{K}{\nu} + \frac{K}{P_\nu} \leqslant \frac{K}{P_\nu};$$

(c) $$\sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} \leqslant \frac{K}{\nu} \leqslant \frac{K}{P_\nu}.$$

(a) is given in Ahmad and Khan [1974, Lemma 3(c)] and (b) and (c) in Ahmad (1966, Lemmas 3 and 1).

√ *Lemma* 4 (Ahmad and Khan 1974, Lemma 4) — If $p_0 > 0$ and $\{p_n\}$ is non-negative and non-increasing, then for $k \geqslant \nu$ (ν finite), $\nu \geqslant 1$ and $\mu \geqslant 1$,

$$\sum_{n=\nu}^{k} \frac{1}{P_{n+\mu}P_{n+\mu-1}} (P_{n+\mu}p_{n-\nu} - P_{n-\nu}p_{n+\mu}) \leqslant \frac{K}{P_{\nu+\mu}}.$$

Lemma 5 (Ahmad and Khan 1974, Lemma 5) — Let $\beta \geqslant 0$. If $p_0 > 0$ and $\{p_n\}$ be nonnegative and non-increasing and if $\epsilon_n = O(n)$, then

$$\sum_{n=0}^{\infty} \frac{N^{\beta}(p;\, n,\, \rho)}{P_{n+\rho}P_{n+\rho-1}} = \Delta^{\beta}(\rho^{-1}\epsilon_\rho).$$

Lemma 6 — If $p_0 > 0$ and $\{p_n\}$ is nonnegative and non-increasing, then for $\nu \geqslant 0$,

$$\sum_{n=\nu}^{\infty} \left| \frac{p_{n-\nu-1}}{P_{n+\rho-1}} - \frac{p_{n-\nu}}{P_{n+\rho}} \right| \leqslant \frac{2p_0}{P_{\nu+\rho}}.$$

This is the correct form of Lemma 11 of Ahmad and Khan (1974). The proof is easy.

§5. *Proof of theorem* 1 — We have

$$\tau_n - \tau_{n-1} = \frac{1}{P_nP_{n-1}} \sum_{\nu=1}^{n} (P_np_{n-\nu} - P_{n-\nu}p_n)\, \epsilon_\nu a_\nu$$

$$= \frac{1}{P_nP_{n-1}} \sum_{\nu=1}^{n} (P_np_{n-\nu} - P_{n-\nu}p_n) \frac{\epsilon_\nu}{\nu} \sum_{\rho=1}^{\nu} A_{\nu-\rho}^{-\beta-1} A_\rho^{\beta}\, t_\rho^{\beta}$$

$$= \sum_{\rho=1}^{n} \frac{\rho A_\rho^{\beta}}{\lambda_\rho P_nP_{n-1}} N^{\beta}(p;\, n-\rho,\, \rho) \frac{t_\rho^{\beta}}{\rho} \lambda_\rho$$

$$= \sum_{\rho=1}^{n} S_{n,\rho} u_\rho$$

where

$$S_{n,\rho} = \frac{\rho A_\rho^{\beta}}{\lambda_\rho P_nP_{n-1}} N^{\beta}(p;\, n-\rho,\, \rho)$$

and

$$u_\rho = \rho^{-1}\lambda_\rho t_\rho^\beta .$$

Hence the necessary and sufficient condition for $\Sigma \mid \tau_n - \tau_{n-1} \mid < \infty$, whenever $\Sigma\, n^{-1}\lambda_n \mid t_n^\beta \mid < \infty$, by Lemma 1, is

$$\sum_{n=1}^{\infty} \mid S_{n,\rho} \mid = \sum_{n=\rho}^{\infty} \frac{\rho A_\rho^\beta}{\lambda_\rho P_n P_{n-1}} \mid N^\beta(p; n-\rho, \rho) \mid < \infty,$$

or

$$M(\rho) = \sum_{n=0}^{\infty} \frac{1}{P_{n+\rho} P_{n+\rho-1}} \mid N^\beta(p; n, \rho) \mid = O(\rho^{-\beta-1}\lambda_\rho). \qquad ...(5.1)$$

Necessity : The condition (i) is necessary. For, since the first term in $M(\rho)$ is

$$\frac{1}{P_\rho P_{\rho-1}} \mid N^\beta(p; 0, \rho) \mid$$

we have

$$\frac{1}{P_\rho P_{\rho-1}} \cdot \frac{p_0 P_{\rho-1}}{\rho} \mid \epsilon_\rho \mid = O(\rho^{-\beta-1}\lambda_\rho)$$

i.e. $\qquad \mid \epsilon_\rho \mid = O(\rho^{-\beta} P_\rho \lambda_\rho), \quad \text{as} \quad p_0 > 0.$

The condition (ii) is necessary. For, since the condition (i) implies that $\epsilon_\rho = O(\rho)$, and (5.1) together with Lemma 5 implies

$$\triangle^\beta(\rho^{-1}\epsilon_\rho) = \sum_{n=0}^{\infty} \frac{N^\beta(p; n, \rho)}{P_{n+\rho} P_{n+\rho-1}},$$

hence

$$\mid \triangle^\beta(\rho^{-1}\epsilon_\rho) \mid \leqslant \sum_{n=0}^{\infty} \frac{\mid N^\beta(p; n, \rho) \mid}{P_{n+\rho} P_{n+\rho-1}} = O(\rho^{-\beta-1}\lambda_\rho).$$

Sufficiency : Suppose that conditions (i) and (ii) hold. Then it is sufficient to prove that (5.1) holds good.

Case I : *When* $\beta = 1$ — Since $A_\nu^{-2} = 1$ (when $\nu = 0$) and $= -1$ (when $\nu = 1$) and it is zero for all other values of ν, then

$$N^1(p; n, \rho) = \sum_{\nu=0}^{n} A_\nu^{-2} (P_{n+\rho}P_{n-\nu} - P_{n-\nu}P_{n+\rho}) \frac{\epsilon_{\nu+\rho}}{(\nu+\rho)}$$

$$= (P_{n+\rho}p_n - P_n p_{n+\rho}) \frac{\epsilon_\rho}{\rho} - (P_{n+\rho}p_{n-1} - P_{n-1}p_{n+\rho}) \frac{\epsilon_{\rho+1}}{(\rho+1)}$$

$$= (P_{n+\rho}p_n - P_n p_{n+\rho}) \left(\frac{\epsilon_\rho}{\rho} - \frac{\epsilon_{\rho+1}}{\rho+1}\right)$$

$$+ [(P_{n+\rho}p_n - P_n p_{n+\rho}) - (P_{n+\rho}p_{n-1} - P_{n-1}p_{n+\rho})] \frac{\epsilon_{\rho+1}}{(\rho+1)}$$

$$= (P_{n+\rho}p_n - P_n p_{n+\rho}) \triangle \left(\frac{\epsilon_\rho}{\rho}\right) - P_{n+\rho} \triangle p_{n-1} \frac{\epsilon_{\rho+1}}{\rho+1}$$

$$- p_n p_{n+\rho} \frac{\epsilon_{\rho+1}}{\rho+1}$$

and hence

$$\sum_{n=0}^{\infty} \frac{|N^1(p, n, \rho)|}{P_{n+\rho}P_{n+\rho-1}} \leqslant \sum_{n=0}^{\infty} \frac{(P_{n+\rho}p_n - P_n p_{n+\rho})}{P_{n+\rho}P_{n+\rho-1}} \left| \triangle \left(\frac{\epsilon_\rho}{\rho}\right)\right|$$

$$+ \sum_{n=0}^{\infty} \frac{|\triangle p_{n-1}|}{P_{n+\rho-1}} \frac{|\epsilon_{\rho+1}|}{(\rho+1)} + \sum_{n=0}^{\infty} \frac{p_n p_{n+\rho}}{P_{n+\rho}P_{n+\rho-1}} \frac{|\epsilon_{\rho+1}|}{\rho+1}$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say}$$

where

$$\sum\nolimits_1 = |\triangle (\rho^{-1}\epsilon_\rho)| \sum_{n=\rho}^{\infty} \frac{(P_n p_{n-\rho} - P_{n-\rho} p_n)}{P_n P_{n-1}}$$

$$= O(|\triangle(\rho^{-1}\epsilon_\rho)|) \qquad \text{(by Lemma 3(a))}$$

$$= O(\rho^{-2}\lambda_\rho) \qquad \text{(by condition (ii))};$$

$$\sum\nolimits_2 = \frac{|\epsilon_{\rho+1}|}{\rho+1} \sum_{n=\rho}^{\infty} \frac{|\triangle p_{n-\rho-1}|}{P_{n-1}}$$

$$\leqslant K \cdot \frac{1}{P_\rho} \cdot \frac{|\epsilon_{\rho+1}|}{\rho+1} \qquad \text{(by Lemma 3(b))}$$

$$\leqslant K \frac{1}{P_\rho} \cdot \frac{P_\rho}{\rho} \cdot \frac{\lambda_\rho}{\rho+1}$$

$$= O(\rho^{-2}\lambda_\rho) \qquad \text{(by condition (i))};$$

and

$$\sum\nolimits_3 = \frac{|\epsilon_{\rho+1}|}{\rho+1}\sum_{n=\rho}^{\infty}\frac{p_n p_{n-\rho}}{P_n P_{n-1}} \leqslant K\,.\,\frac{1}{P_\rho}\cdot\frac{|\epsilon_{\rho+1}|}{\rho+1} \qquad \text{(by Lemma 3(c))}$$

$$\leqslant K\,.\,\frac{1}{P_\rho}\cdot\frac{P_\rho}{\rho(\rho+1)}\lambda_\rho$$

$$= O(\rho^{-2}\lambda_\rho) \qquad \text{(by condition (i)).}$$

Case II: *When* $0<\beta<1,\ \beta>1$ — Let us write

$$M(\rho) = \left(\sum_{n=0}^{k} + \sum_{n=k+1}^{\infty}\right)\left(\frac{1}{P_{n+\rho}P_{n+\rho-1}}\right)|N^\beta(p;n,\rho)|$$

$$= M_1(\rho) + M_2(\rho), \text{ say.} \qquad \ldots(5.2)$$

Then, since

$$M_1(\rho) = \sum_{n=0}^{k}\frac{|N^\beta(p;n,\rho)|}{P_{n+\rho}P_{n+\rho-1}}$$

$$= \sum_{n=0}^{k}\frac{1}{P_{n+\rho}P_{n+\rho-1}}\left|\sum_{\nu=0}^{n}A_\nu^{-\beta-1}(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho})\frac{\epsilon_{\nu+\rho}}{(\nu+\rho)}\right|$$

$$\leqslant \sum_{n=0}^{k}\frac{1}{P_{n+\rho}P_{n+\rho-1}}\sum_{\nu=0}^{n}|A_\nu^{-\beta-1}|\,.\,(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho})\frac{|\epsilon_{\nu+\rho}|}{(\nu+\rho)}$$

$$= \sum_{\nu=0}^{k}|A_\nu^{-\beta-1}|\frac{|\epsilon_{\nu+\rho}|}{\nu+\rho}\sum_{n=\nu}^{k}\frac{(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho})}{P_{n+\rho}P_{n+\rho-1}}$$

$$\leqslant K\sum_{\nu=0}^{k}|A_\nu^{-\beta-1}|\frac{P_{\nu+\rho}}{(\nu+\rho)^\beta}\cdot\frac{\lambda_{\nu+\rho}}{(\nu+\rho)}\cdot\frac{1}{P_{\nu+\rho}}$$

(by Lemma 4 and condition (i))

$$= O(\rho^{-\beta-1}\lambda_\rho),$$

by hypothesis, it remains to prove that

$$M_2(\rho) = O(\rho^{-\beta-1}\lambda_\rho).$$

We use the identity

$$\nu A_\nu^{-\beta-1} = \beta A_\nu^{-\beta-1} - \beta A_\nu^{-\beta}$$

to write

$$N^\beta(p; n, \rho) = (\rho + \beta)\, \bar{N}^\beta(p; n, \rho) - \beta \bar{N}^{\beta-1}(p; n, \rho)$$

so that

$$M_2(\rho) \leqslant (\rho + \beta)\, M_2^\beta(\rho) + \beta M_2^{\beta-1}(\rho)$$

where

$$M_2^\gamma(\rho) = \sum_{n=k+1}^{\infty} \frac{|\bar{N}^\gamma(p; n, \rho)|}{P_{n+\rho} P_{n+\rho-1}}, \qquad (\gamma = \beta, \beta - 1).$$

Hence, it is sufficient, to prove that

$$M_2^\gamma(\rho) = O(\rho^{-\gamma-2}\lambda_\rho), \qquad (\gamma = \beta, \beta - 1). \qquad \ldots(5.3)$$

Proof of (5.3) *When* $\gamma = \beta - 1$ — For $0 < \beta < 1$,

$$M_2^{\beta-1}(\rho) = \sum_{n=k+1}^{\infty} \frac{|\bar{N}^{\beta-1}(p; n, \rho)|}{P_{n+\rho} P_{n+\rho-1}}$$

$$< \sum_{n=0}^{\infty} \frac{1}{P_{n+\rho} P_{n+\rho-1}} \left(\sum_{\nu=0}^{n} A_\nu^{-\beta} (P_{n+\rho} p_{n-\nu} - P_{n-\nu} p_{n+\rho}) \frac{|\epsilon_{\nu+\rho}|}{(\nu + \rho)^2} \right)$$

$$= \sum_{n=\rho}^{\infty} \left| \sum_{\nu=\rho}^{n} A_{\nu-\rho}^{-\beta} \frac{(P_n p_{n-\nu} - P_{n-\nu} p_n)}{P_n P_{n-1}} \cdot \frac{|\epsilon_\nu|}{\nu^2} \right|$$

$$= \sum_{\nu=\rho}^{\infty} A_{\nu-\rho}^{-\beta} \frac{|\epsilon_\nu|}{\nu^2} \sum_{n=\nu}^{\infty} \frac{(P_n p_{n-\nu} - P_{n-\nu} p_n)}{P_n P_{n-1}}$$

$$\leqslant K \sum_{\nu=\rho}^{\infty} A_{\nu-\rho}^{-\beta} \frac{|\epsilon_\nu|}{\nu^2} \qquad \text{(by Lemma 3(a))}$$

$$\leqslant K \sum_{\nu=\rho}^{\infty} \frac{A_{\nu-\rho}^{-\beta}}{\nu} \frac{P_\nu}{\nu^\beta} \frac{\lambda_\nu}{\nu} \qquad \text{(by condition (i))}$$

$$\leqslant K \frac{\lambda_\rho}{\rho} \frac{1}{\rho A_\rho^{\beta-1}}$$

$$= O(\rho^{-\beta-1}\lambda_\rho),$$

by hypothesis and Lemma 2; and for $\beta > 1$, replacing $A_\nu^{-\beta}$ by $| A_\nu^{-\beta} |$ in the preceding analysis, again by hypothesis and Lemma 3(a), we have

$$M_2^{\beta-1}(\rho) \leqslant K \sum_{\nu=\rho}^{\infty} | A_{\nu-\rho}^{-\beta} | \cdot \frac{P_\nu}{\nu^\beta} \cdot \frac{\lambda_\nu}{\nu^2}$$

$$\leqslant K \frac{\lambda_\rho}{\rho^\beta \cdot \rho} \cdot \sum_{\nu=\rho}^{\infty} | A_{\nu-\rho}^{-\beta} |$$

$$= O(\rho^{-\beta-1}\lambda_\rho),$$

by using the fact that $\Sigma \, | A_n^\alpha | < \infty$, when $\alpha < -1$.

Proof of (5.3) *when* $\gamma = \beta$ — Since

$$\bar{N}^\beta(p; n, \rho) = \sum_{\nu=0}^{n} A_\nu^{-\beta-1} (P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho}) \, \epsilon'_{\nu+\rho}$$

$$= \sum_{\nu=0}^{n} \Delta_\nu(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho}) \sum_{j=0}^{\nu} A_j^{-\beta-1} \, \epsilon'_{j+\rho}$$

$$+ (P_{n+\rho}p_{n-\nu-1} - P_{n-\nu-1}p_{n+\rho})_{\nu=n} \sum_{j=0}^{n} A_j^{-\beta-1} \, \epsilon'_{j+\rho}$$

$$= \sum_{\nu=0}^{n} \Delta_\nu(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho}) \sum_{j=0}^{\nu} A_j^{-\beta-1} \, \epsilon'_{j+\rho}$$

$$= \bar{N}_1^\beta(p; n, \rho) + \bar{N}_2^\beta(p; n, \rho) + \bar{N}_3^\beta(p; n, \rho),$$

where

$$\epsilon'_{j+\rho} = \epsilon_{j+\rho}/(j+\rho)^2,$$

$$\bar{N}_1^\beta(p; n, \rho) = (P_{n+\rho}p_n - P_n p_{n+\rho}) \, \Delta^\beta(\epsilon'_\rho),$$

$$\bar{N}_2^\beta(p; n, \rho) = -(P_{n+\rho}p_n - P_n p_{n+\rho}) \sum_{j=n+1}^{\infty} A_j^{-\beta-1} \, \epsilon'_{j+\rho},$$

$$\bar{N}_3^\beta(p; n, \rho) = -\sum_{\nu=0}^{n} \Delta_\nu(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho}) \sum_{j=\nu+1}^{n} A_j^{-\beta-1} \, \epsilon'_{j+\rho}.$$

In order to prove (5.3) for $\gamma = \beta$, it is sufficient to show that, for $0 < \beta < 1$ and $\beta > 1$,

$$\sum\nolimits_i = \sum_{n=k+1}^{\infty} \frac{| \bar{N}_i^{\beta} (p; n, \rho) |}{P_{n+\rho} P_{n+\rho-1}} = O(\rho^{-\beta-2}\lambda_\rho), \qquad (i = 1, 2, 3). \quad ...(5.4)$$

Proof of (5.4) *when* $i = 1$ — We have

$$\sum\nolimits_1 = \sum_{n=k+1}^{\infty} \frac{| \bar{N}_1^{\beta} (p; n, \rho) |}{P_{n+\rho} P_{n+\rho-1}}$$

$$\leqslant \sum_{n=0}^{\infty} \frac{(P_{n+\rho} p_n - P_n p_{n+\rho})}{P_{n+\rho} P_{n+\rho-1}} \, | \triangle^{\beta}(\epsilon'_\rho) |$$

$$= | \triangle^{\beta} \epsilon'_\rho | \sum_{n=\rho}^{\infty} \frac{(P_n p_{n-\rho} - P_{n-\rho} p_n)}{P_n P_{n-1}}$$

$$= O(\rho^{-\beta-2}\lambda_\rho),$$

by hypothesis and condition (ii) and Lemma 3(a).

Proof of (5.4) *when* $i = 2$ — We have

$$\sum\nolimits_2 \leqslant \sum_{n=0}^{\infty} \frac{(P_{n+\rho} p_n - P_n p_{n+\rho})}{P_{n+\rho} P_{n+\rho-1}} \sum_{j=n+1}^{\infty} | A_j^{-\beta-1} | \; | \epsilon'_{j+\rho} |$$

$$= \sum_{j=1}^{\infty} | A_j^{-\beta-1} | \; | \epsilon'_{j+\rho} | \sum_{n=\rho}^{\rho+j-1} \frac{P_n p_{n-\rho} - P_{n-\rho} p_n}{P_n P_{n-1}}$$

$$= \sum_{j=1}^{\infty} | A_j^{-\beta-1} | \cdot | \epsilon'_{j+\rho} | \sum_{n=\rho}^{\rho+j-1} \left(\frac{P_{n-\rho}}{P_n} - \frac{P_{n-\rho-1}}{P_{n-1}} \right)$$

$$= \sum_{j=1}^{\infty} | A_j^{-\beta-1} | \cdot | \epsilon'_{j+\rho} | \frac{P_{j-1}}{P_{\rho+j-1}}$$

$$\leqslant \sum_{j=1}^{\infty} | A_j^{-\beta-1} | (j + \rho)^{-\beta} . P_{j+\rho} \frac{\lambda_{j+\rho}}{(j + \rho)^2} \frac{P_{j-1}}{P_{j+\rho-1}}$$

$$= O(\rho^{-\beta-2}\lambda_\rho) \sum_{j=1}^{\infty} | A_j^{-\beta-1} | P_{j-1}$$

$$= O(\rho^{-\beta-2}\lambda_\rho)$$

since $\{p_k\}$ is non-increasing, $P_k \leqslant (k+1) p_0 = O(k)$ and hence $\sum_{j=1}^{\infty} | A_j^{-\beta-1} | P_{j-1}$ converges.

Proof of (5.4) *when* $i = 3$ — We have

$$\sum\nolimits_3 < \sum_{n=0}^{\infty} \frac{1}{P_{n+\rho}P_{n+\rho-1}} \left| \sum_{\nu=0}^{n} \Delta_\nu(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho}) \sum_{j=\nu+1}^{\infty} A_j^{-\beta-1}\epsilon'_{j+\rho} \right|$$

$$\leqslant \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} | A_j^{-\beta-1} | \cdot | \epsilon'_{j+\rho} | \sum_{n=\nu}^{\infty} \frac{| \Delta_\nu(P_{n+\rho}p_{n-\nu} - P_{n-\nu}p_{n+\rho}) |}{P_{n+\rho}P_{n+\rho-1}}$$

$$= \sum_{j=1}^{\infty} | A_j^{-\beta-1} | \, | \epsilon'_{j+\rho} | \sum_{\nu=0}^{j-1} \sum_{n=\nu}^{\infty} \left| \frac{p_{n-\nu-1}}{P_{n+\rho-1}} - \frac{p_{n-\nu}}{P_{n+\rho}} \right|$$

$$= \sum_{j=1}^{\infty} | A_j^{-\beta-1} | \, | \epsilon'_{j+\rho} | \sum_{\nu=0}^{j-1} \frac{2p_0}{P_{\nu+\rho}}$$

$$\leqslant \frac{2p_0}{P_\rho} \sum_{j=1}^{\infty} | A_j^{-\beta-1} | \, j \, \frac{P_{j+\rho}}{(j+\rho)^\beta} \frac{\lambda_{j+\rho}}{(j+\rho)^2}$$

$$= O(\rho^{-\beta-2}\lambda_\rho)$$

since, as in the proof of $(5.4)_2$, $\sum_{j=1}^{\infty} | A_j^{-\beta-1} | \, j < \infty$.

This terminates the proof of the sufficiency part of the theorem and hence the theorem is proved.

§6. Since $\left| N, \frac{1}{n+1} \right| \subset | C, \beta |$ for every $\beta > 0$, by taking $\lambda_n = 1$ for all n and $p_n = (n+1)^{-1}$ for $n \geqslant 0$, we obtain the following harmonic summability factor theorem.

Theorem 2 — The necessary and sufficient conditions that the series $\Sigma\, a_n\epsilon_n$ should be summable $\left| N, \frac{1}{n+1} \right|$, whenever $\Sigma\, a_n$ is summable $| C, \beta |$, are:

(i) $\epsilon_n = O(n^{-\beta} \log n)$;

(ii) $\Delta^{\beta}(n^{-1}\epsilon_n) = O(n^{-\beta-1})$.

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