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THE NUMBER OF M-VOID DIVISORS OF AN INTEGER

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Let M be a set of positive integers with minimal element $\geqslant 2$. A positive integer n is called M-void if each canonical exponent in the factorization of n is outside M. Denoting by $\tau_M(n)$ the number of divisors of n which are M-void, in this paper, we establish an asymptotic formula for $\sum_{n \leqslant x} \tau_M(n)$.

1. INTRODUCTION

Let M be a set of positive integers with minimal element $r \geqslant 2$. A positive integer n is called M-void if in the canonical factorisation of n into product of prime powers, each exponent lies outside M. The integer 1 is also considered to be M-void. The concept of an M-void integer was introduced by Rieger (1973). A divisor $d > 0$ of the positive integer n will be called an M-void divisor of n if d is an M-void integer. Let $\tau_M(n)$ denote the number of M-void divisors of n. Writing $S = \{n \mid n \text{ is integral} \geqslant r, n \notin M\}$, we put $k \equiv k_M = \min S$ or ∞ according as S is nonempty or not. In this paper, we prove the following:

Theorem 1 — For $x \geqslant 1$,

$$\sum_{n \leqslant x} \tau_M(n) = \alpha_M x \left(\log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)} + \frac{k\zeta'(k)}{\zeta(k)} + \frac{f'_M(1)}{f_M(1)} \right) + \Delta_M(x) \qquad \text{...(1.1)}$$

where $\quad \Delta_M(x) = O_M(x^{1/r} \exp\{- A (\log 2x)^{3/5} (\log\log 3x)^{-1/5}\})$

or $O_M(x^{\alpha})$ according as $r = 2, 3$ or $r \geqslant 4$, A is a positive constant, γ is Euler's constant, α is the number which appears in the Dirichlet divisor problem (2.3) $f(s)$ is given by (2.6) and α_M is the constant given by (2.7).

Theorem 2 — If the Riemann hypothesis is true, then for $x \geqslant 1$, the error term $\Delta_M(x)$ in (1.1) is given by $\Delta_M(x) = O_M(x^{(2-\alpha)/(1+2r(1-\alpha))}\omega(x))$ or $O_M(x^{\alpha})$ according as $r = 2, 3$ or $r \geqslant 4$ where $\omega(x) = \exp\{B \log 2x (\log\log 3x)^{-1}\}$, B being a positive constant and α is the number which appears in the Dirichlet divisor problem.

The auxiliary results needed for the proofs of Theorems 1 and 2 are given in section 2 while in section 3, the proofs of Theorems 1 and 2 are carried out. By specializing the set M, we deduce, in section 4, results due to Rao and Suryanarayana

(1973), Suryanarayana and Rao (1976) and Suryanarayana and Prasad (1977) concerning the divisor functions respectively associated with (k, r)-integers, unitarily r-free integers and semi-r-free integers.

2. Preliminaries

Let Q_M denote the set of all M-void integers and q_M, the characteristic function of Q_M. In case, $M = M_r = \{r, r+1, r+2, ...\}$ we write Q_r, q_r, $\tau_{(r)}(n)$ to denote respectively Q_{M_r}, q_{M_r} and $\tau_{M_r}(n)$. It may be noted that Q_r is the well-known set of all r-free integers. We denote by $g_M(n)$ the unique arithmetical function determined by $q_M(n) = \sum_{d\delta=n} g_M(d)\, q_r(\delta)$. We need the following lemmas:

Lemma 2.1 (Suryanarayana and Prasad 1971, Theorem 3.1) — For $x \geqslant 1$,

$$\sum_{n\leqslant x} \tau_{(r)}(n) = \frac{x}{\zeta(r)} \left\{\log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)}\right\} + \Delta_{(r)}(x) \qquad ...(2.1)$$

where $\Delta_{(r)}(x) = O_r(x^{1/r}\delta(x))$ or $O_r(x^\alpha)$ according as $r = 2, 3$ or $r \geqslant 4$; $\delta(x)$ being given by

$$\delta(x) = \exp\{-A(\log 2x)^{3/5}(\log\log 3x)^{-1/5}\} \qquad ...(2.2)$$

A being a positive constant and α is the number which appears in the Dirichlet divisor problem, namely

$$\sum_{n\leqslant x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^\alpha) \qquad ...(2.3)$$

where $\tau(n)$ is the number of positive divisors of n.

Remark 1 : It is known that $\frac{1}{4} < \alpha < \frac{1}{3}$ (Hardy and Wright 1960, p. 272). The best result known to date is due to Kolesnik (1973, p. 28) who proved that the error term in (2.3) is $O(x^{(346+1047\epsilon)/1047})$ for each $\epsilon > 0$. There is a conjecture that $\alpha = \frac{1}{4} + \epsilon$.

Lemma 2.2 (Suryanarayana and Prasad 1971, Theorem 3.2) — If the Riemann Hypothesis is true, then for $x \geqslant 1$, the error term $\Delta_{(r)}(x)$ in case $r = 2$ and 3 is given by

$$\Delta_{(r)}(x) = O_r\left(x^{\frac{2-\alpha}{1+2r(1-\alpha)}}\,\omega(x)\right)$$

where

$$\omega(x) = \exp\{B\log 2x(\log\log 3x)^{-1}\} \qquad ...(2.4)$$

B being a positive constant and α is given by (2.3).

Lemma 2.3 — For $s > 1$,

$$\sum_{n=1}^{\infty} \frac{q_M(n)}{n^s} = \frac{\zeta(s)\,\zeta(ks)}{\zeta(rs)} f(s) \qquad \ldots(2.5)$$

where

$$f(s) = f_M(s) \equiv \prod_p \left\{1 - p^{-2ks} + \frac{1 - p^{-ks}}{1 - p^{-rs}} \times \left(p^{-(k+r)s} - (1 - p^{-s}) \sum_{k<a\in M} p^{-as}\right)\right\} \qquad \ldots(2.6)$$

the product ranging over all primes p.

PROOF: Since $q_M(n) = 1$ or 0 according as $n \in Q_M$ or not, the series $\sum_{n=1}^{\infty} q_M(n)\, n^{-s}$ converges absolutely for $s > 1$. Also the general term is a multiplicative function of n so that the series could be expanded into an infinite product of Euler type (Hardy and Wright 1960, Theorem 286). Thus for $s > 1$

$$\begin{aligned}
\sum_{n=1}^{\infty} q_M(n)\, n^{-s} &= \prod_p \left\{ \sum_{m=0}^{\infty} p^{-ms} - \sum_{a\in M} p^{-as}\right\} \\
&= \zeta(s) \prod_p \left\{1 - (1 - p^{-s}) \sum_{a\in M} p^{-as}\right\} \\
&= \frac{\zeta(s)}{\zeta(rs)} \prod_p \left\{1 + \left(p^{-ks} - (1-p^{-s}) \sum_{k<a\in M} p^{-as}\right)\left(1 - p^{-rs}\right)^{-1}\right\} \\
&= \frac{\zeta(s)\,\zeta(ks)}{\zeta(rs)} \prod_p \left\{\left(1 + \frac{p^{-ks} - (1 - p^{-s}) \sum_{k<a\in M} p^{-as}}{1 - p^{-rs}}\right) \times (1 - p^{-ks})\right\} \\
&= \frac{\zeta(s)\,\zeta(ks)}{\zeta(rs)} \prod_p \left\{1 - p^{-2ks} + \frac{1 - p^{-ks}}{1 - p^{-rs}} \times \left(p^{-(k+r)s} - (1 - p^{-s}) \sum_{k<a\in M} p^{-as}\right)\right\}.
\end{aligned}$$

This proves Lemma 2.3.

Remark 2 : From the above argument, we see that the products defining $\frac{\zeta(ks)}{\zeta(rs)} f_M(s)$ and $f_M(s)$ converge absolutely for $s > 1/r$. Hence in particular

$$\alpha_M \equiv \frac{\zeta(k)}{\zeta(r)} f_M(1) = \prod_p \left\{1 - (1 - p^{-1}) \sum_{a \in M} p^{-a}\right\}. \qquad \text{...(2.7)}$$

Lemma 2.4 (Estermann 1952; Theorem 41) — If $f(n)$ is multiplicative and $\prod_p \{ \sum_{m=0}^{\infty} | f(p^m) | \}$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges absolutely and

$$\sum_{n=1}^{\infty} f(n) = \prod_p \{ \sum_{m=0}^{\infty} f(p^m)\}.$$

Lemma 2.5 — For $x \geqslant 1$ and each $\epsilon > 0$,

$$\sum_{n \leqslant x} | g_M(n) | = O_{M,\epsilon}(x^{(1/k)+\epsilon}) \qquad \text{...(2.8)}$$

and

$$\sum_{n>x} \frac{| g_M(n) |}{n} = O_{M,\epsilon}(x^{-1+(1/k)+\epsilon}). \qquad \text{...(2.9)}$$

PROOF : We note that the infinite product appearing on the right of (2.6), defining $f(s)$, converges absolutely for $s > 1/k$. Also, by taking

$$M = M_r = \{r, r + 1, ...\}$$

in Lemma 2.3, we obtain, for $s > 1$

$$\sum_{n=1}^{\infty} q_r(n)\, n^{-s} = \zeta(s)/\zeta(rs).$$

But by the definition of g_M, we have, for $s > 1$

$$\sum_{n=1}^{\infty} g_M(n)\, n^{-s} \sum_{n=1}^{\infty} q_r(n)\, n^{-s} = \sum_{n=1}^{\infty} q_M(n)\, n^{-s}.$$

Hence by Lemma 2.3, for $s > 1$

$$\sum_{n=1}^{\infty} g_M(n)\, n^{-s} = \zeta(ks)\, f(s)$$

$$= \prod_p \{1 + p^{-ks} + (1 - p^{-rs})^{-1} (p^{-(k+r)s} - (1 - p^{-s}) \sum_{k<a \in M} p^{-as})\}. \qquad \text{...(2.10)}$$

Since the infinite product appearing on the right of above converges absolutely for $s > 1/k$, it follows from Lemma 2.4 that the series $\sum_{n=1}^{\infty} g_M(n)\, n^{-s}$ converges absolutely for $s > 1/k$. Using this, we obtain (2.8) and (2.9) in a routine way by appealing to the theorem of partial summation.

Lemma 2.6 — For $s > 1/k$,

$$\sum_{n=1}^{\infty} \frac{g_M(n) \log n}{n^s} = - \{k\zeta'(ks)\, f(s) + f'(s)\, \zeta(ks)\} \qquad \ldots(2.11)$$

where $f(s)$ is given by (2.6).

PROOF : This series is uniformly convergent for $s \geqslant (1/k) + \epsilon > 1/k$ and so by termwise differentiation of the series in (2.10) with respect to s, we get (2.11).

3. PROOFS OF THEOREMS 1 AND 2

Let $q_M(n)$ denote the characteristic function of the set Q_M of all M-void integers. Then by the definition of g_M, we have

$$\begin{aligned}\tau_M(n) &= \sum_{rs=n} q_M(r) = \sum_{rs=n} \sum_{d\delta=r} g_M(d)\, q_r(\delta) \\ &= \sum_{d\delta s=n} g_M(d)\, q_r(\delta) = \sum_{du=n} g_M(d) \sum_{\delta s=u} q_r(\delta) \\ &= \sum_{du=n} g_M(d)\, \tau_{(r)}(u).\end{aligned}$$

Hence

$$\sum_{n\leqslant x} \tau_M(n) = \sum_{du\leqslant x} g_M(d)\, \tau_{(r)}(u),$$

the summation on the right being extended over all ordered pairs (d, u) such that $du \leqslant x$.

Using Lemma 2.1, we have

$$\begin{aligned}\sum_{n\leqslant x} \tau_M(n) &= \sum_{d\leqslant x} g_M(d) \Big(\sum_{u\leqslant x/d} \tau_{(r)}(u)\Big) \\ &= \sum_{d\leqslant x} g_M(d) \left\{\frac{x}{d\zeta(r)} \left(\log \frac{x}{d} + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)}\right)\right. \\ &\qquad \left. + \Delta_{(r)}\left(\frac{x}{d}\right)\right\}\end{aligned}$$

(*equation continued on p.* 754)

$$= \frac{x}{\zeta(r)}\left(\log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)}\right)\sum_{d=1}^{\infty}\frac{g_M(d)}{d}$$

$$-\frac{x}{\zeta(r)}\sum_{d=1}^{\infty}\frac{g_M(d)\log d}{d} + O\left(x\log 2x\sum_{d<x}\frac{|g_M(d)|}{d}\right)$$

$$+ O\left(x\sum_{d>x}\frac{|g_M(d)|\log d}{d}\right) + \sum_{d\leqslant x} g_M(d)\,\Delta_{(r)}\left(\frac{x}{d}\right). \qquad \ldots(3.1)$$

The first O-term above is $O_{M,\epsilon}(x\log 2x(x^{(\epsilon k-2k+2)/2k})) = O_{M,\epsilon}(x^{(1+\epsilon k)/k})$ and the second O-term is $O_\epsilon\left(x\log 2x(x^{(\epsilon k-2k+2)/2k})\sum_{d>x}\frac{|g_M(d)|}{d^{(2+\epsilon k)/2k}}\right) = O_{M,\epsilon}(x^{(1+\epsilon k)/k})$ by restricting ϵ to satisfy $0<\epsilon<\frac{1}{2}(r^{-1}-k^{-1})$ and using (2.9). Since for fixed $\epsilon>0$, $x^\epsilon\delta(x)$ is monotonically increasing for large x, we have by Lemma 2.1, in case $r=2$ and 3

$$\sum_{d\leqslant x} g_M(d)\,\Delta_{(r)}(x/d) = O_r(\sum_{d\leqslant x}|g_M(d)|\,(x/d)^{1/r}\,\delta(x/d))$$

$$= O_r(\sum_{d\leqslant x}|g_M(d)|\,(x/d)^{(1-\epsilon r)/r}\,(x/d)^\epsilon\,\delta(x/d))$$

$$= O_{r,\epsilon}(x^{1/r}\delta(x)\sum_{d\leqslant x}|g_M(d)|\,d^{-(1/r)+\epsilon})$$

$$= O_{M,\epsilon}(x^{1/r}\delta(x))$$

since $\epsilon<\frac{1}{2}(r^{-1}-k^{-1})$. In case $r\geqslant 4$, we have

$$\sum_{d\leqslant x} g_M(d)\,\Delta_{(r)}(x/d) = O_r(\sum_{d\leqslant x}|g_M(d)|\,(x/d)^\alpha)$$

$$= O_r(x^\alpha\sum_{d\leqslant x}|g_M(d)|\,d^{-\alpha})$$

$$= O_M(x^\alpha)$$

since $\alpha>\frac{1}{4}\geqslant\frac{1}{r}>\frac{1}{k}$. Instead of using lemma 2.1, if we use Lemma 2.2 in case $r=2$ and 3, we obtain

$$\sum_{d\leqslant x} g_M(d)\,\Delta_{(r)}(x/d) = O_r(\sum_{d\leqslant x}|g_M(d)|\,(x/d)^{(2-\alpha)/(1+2r(1-\alpha))}\omega(x/d))$$

$$= O_r\left(x^{(2-\alpha)/(1+2r(1-\alpha))}\,\omega(x)\sum_{d\leqslant x}\frac{|g_M(d)|}{d^{(2-\alpha)/(1+2r(1-\alpha))}}\right)$$

$$= O_M(x^{(2-\alpha)/(1+2r(1-\alpha))}\omega(x))$$

since $\omega(x)$ is increasing for large x and $\frac{2-\alpha}{1+2r(1-\alpha)} > \frac{1}{r+1} \geqslant \frac{1}{k}$. Thus the proofs of Theorems 1 and 2 are complete on noting that

$$\sum_{d=1}^{\infty} g_M(d)\, d^{-1} = \zeta(k) f(1), - \sum_{d=1}^{\infty} g_M(d) (\log d)\, d^{-1} = k\zeta'(k) f(1)$$

$$+ f'(1)\,\zeta(k) \quad \text{and} \quad \alpha_M = \frac{\zeta(k)}{\zeta(r)} f_M(1).$$

4. Applications

In this section, we illustrate Theorems 1 and 2, by specializing the set M. Let r, k be integers $\geqslant 2$ and s a positive integer. We write

$$M^{(1)}(k, r) = \left\{ \begin{matrix} n \mid n \geqslant r,\ n \text{ is congruence to at least one} \\ \text{of } r, r+1, \ldots, k-1 \pmod{k} \end{matrix} \right\} \text{ for } r < k;$$

$$M^{(2)}(s, r) = \{r, 2r, \ldots, sr\};$$

$$M^{(3)}(r) = \{r, 2r, \ldots\};$$

$$M^{(4)}(r) = \{r\}.$$

The sets $Q_M(1)_{(k,r)}, \ldots, Q_M(4)_{(r)}$ will be denoted respectively by $Q_{k,r}$, $Q^*_{s,r}$, Q^*_r, Q^{s*}_r and elements of these will be referred to respectively as (k, r)-integers (Rao and Harris 1966 and Cohen 1963), unitarily (s, r)-integers, unitarily r-free integers (Cohen 1961 and 1964), Semi-r-free integers (Suryanarayana 1971). It may be noted that $Q^{s*}_r = Q^*_{1,r}$ and thus the notion of a semi-r-free integer is essentially contained in the works of Cohen (1961 and 1964). Further, for each positive integer n, we write $\tau_{(k,r)}(n)$, $\tau^*_{(s,r)}(n)$, $\tau^*_{(r)}(n)$, $\tau^{s*}_{(r)}(n)$ to mean respectively

$$\tau_M(1)_{(k,r)}(n),\ \tau_M(2)_{(s,r)}(n),\ \tau_M(3)_{(r)}(n),\ \tau_M(4)_{(r)}(n).$$

Taking $M = M^{(1)}(k, r)$, $M^{(3)}(r)$ and $M^{(4)}(r)$ in turn in Theorems 1 and 2, we obtain the following results. In each of these results, γ denotes the Euler's constant, α is the number which appears in the Dirichlet divisor problem (2.3), $\delta(x)$ and $\omega(x)$ are as given in (2.2) and (2.4) respectively.

Corollary 1 (Rao and Suryanarayana 1973, Theorems 1 and 2) — For $x \geqslant 1$

$$\sum_{n \leqslant x} \tau_{(k,r)}(n) = \alpha_{(k,r)} x \left(\log x + 2\gamma - 1 - \frac{r\zeta'(r)}{\zeta(r)} + \frac{k\zeta'(k)}{\zeta(k)} \right) + \Delta_{r,r}(x) \quad \ldots(4.1)$$

where

$$\alpha_{(k,r)} \equiv \alpha_M(1)_{(k,r)} = \frac{\zeta(k)}{\zeta(r)} \qquad ...(4.2)$$

and $\Delta_{k,r}(x) = O_{k,r}(x^{1/r}\delta(x))$ or $O_{k,r}(x^{\alpha})$ according as $r = 2, 3$ or $r \geqslant 4$. Further, on the assumption of the Riemann hypothesis the O-estimate for $\Delta_{k,r}(x)$ in (4.1) could be improved, in case $r = 2$ and 3, to $O_{k,r}(x^{(2-\alpha)/(1+2r(1-\alpha))}\omega(x))$.

Corollary 2 (Suryanarayana and Rao 1976, Theorems 1 and 2) — For $x \geqslant 1$,

$$\sum_{n \leqslant x} \tau^*_{(r)}(n) = \alpha^*_{(r)}\, x \left(\log x + 2\gamma - 1 + r\frac{\zeta'(r)}{\zeta(r)} + \sum_p \frac{(2kp - k - 1)\log p}{p^{k+1} - 2p + 1}\right) + \Delta^*_{(r)}(x) \qquad ...(4.3)$$

where

$$\alpha^*_{(r)} \equiv \alpha_M(3)_{(r)} = \zeta(r) \prod_p \left(1 - \frac{2}{p^r} + \frac{1}{p^{r+1}}\right) \qquad ...(4.4)$$

and $\Delta^*_{(r)}(x) = O_r(x^{1/r}\delta(x))$ or $O_r(x^{\alpha})$ according as $r = 2, 3$ or $r \geqslant 4$. Further, on the assumption of the Riemann hypothesis, the O-estimate for $\Delta^*_{(r)}(x)$ in (4.3) could be improved, in case $r = 2$ and 3, to $O_r(x^{(2-\alpha)/(1+2r(1-\alpha))}\,\omega(x))$.

It should be noted that there is a mistake in the statement of Theorem 1 of Suryanarayana and Rao (1976). In fact, on page 19, line 8 from below there should be $k\zeta'(k)/\zeta(k)$ instead of $\zeta'(k)/\zeta(k)$. A similar correction should be incorporated in line 5 of page 31 of that paper.

Corollary 3 (Suryanarayana and Prasad 1977, Theorems 1 and 2) — For $x \geqslant 1$,

$$\sum_{n \leqslant x} \tau^{**}_{(r)}(n) = \alpha^{**}_{(r)}\, x \left(\log x + 2\gamma - 1 + \sum_p \frac{(rp - r - 1)\log p}{(p^{r+1} - p + 1)}\right) + \Delta^{**}_{(r)}(x) \qquad ...(4.5)$$

where

$$\alpha^{**}_{(r)} \equiv \alpha_M(4)_{(r)} = \prod_p \left(1 - \frac{1}{p^r} + \frac{1}{p^{r+1}}\right) \qquad ...(4.6)$$

and $\Delta^{**}_{(r)}(x) = O_r(x^{1/r}\delta(x))$ or $O_r(x^{\alpha})$ according as $r = 2, 3$ or $r \geqslant 4$. Further, on the assumption of the Riemann hypothesis, the O-estimate for $\Delta^{s*}_{(r)}(x)$ could be improved, in case $r = 2$ and 3, to $O_r(x^{(2-\alpha)/(1+2r(1-\alpha))}\omega(x))$.

Finally, we note that a result similar to the above could be deduced from Theorems 1 and 2 by taking $M = M^{(2)}(s, r)$.

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