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A RADON-NIKODYM THEOREM IN CONNECTION WITH A VECTOR-VALUED INTEGRAL

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An analogue of Radon-Nikodym theorem has been established in the context of measures taking values in a linear lattice and integrals constructed therefrom representing points in some other linear lattice.

1. INTRODUCTION

The prime objective of the present paper is to establish an analogue of the Radon Nikodym theorem in connection with integrals which take on values in linear lattices. To trace the motivation of the present endeavour, one should refer to Kundu and Lahiri (1977, 1979); Kundu and Lahiri (1977) have manufactured a set function ι on a certain σ-algebra $Q(S)$ of subsets of a non-atomic (Halmos 1963) complete Boolean algebra S which takes values in a conditionally complete lattice-ordered-group X starting with a primitive function which satisfies some requirements on H — *a* subalgebra of S. This function ι is countably additive, isotone and satisfies $\iota(\phi) = 0$. We call such a function a measure function. Subsequently, they have discussed a cogent integration theory (see Kundu and Lahiri 1979) with respect to ι and a class of measurable functions which take on values in a linear lattice E; however, the integrals that are constructed lie in some other linear lattice F. The key to this phenomenon lies in a bilinear function $\theta : X \times E \to F$ which is an order-preserving lattice-homomorphism. Its formulation and simple properties have been mentioned without proof in Kundu and Lahiri (1979). Image of any pair (x, e), $x \in X$, $e \in E$ under θ is denoted by xe and we do not distinguish between xe and ex. However, some additional results which had been proved by Kundu and Lahiri (1977) have to be reformulated under weaker situations (vide section 2) which helped to obtain the main result (Theorem 3.5).

With S, $Q(S)$ and ι as described above the ordered set $(S, Q(S), \iota, X)$ shall be called a measure space; elements of $Q(S)$ will be referred to as measurable sets.

To a linear lattice L we adjoin an object ∞ and fit it with the order structure of L by the rule $a < \infty$ for every $a \in L$. For a non-decreasing sequence $\{a_n\}$ in L not bounded above we write 0-$\lim_n a_n = \infty$.

For standard definitions vis-a-vis linear lattice one may refer to Birkhoff (1967), Nakano (1950), Peressini (1967) and the study of measure (outer) function in some other contexts may be seen in Hewitt (1953), Hrycay (1965), Kelley (1969), Maitland (1972), Nakano and Brown (1962) and Sion (1969). However, we state below a number of known definitions for ready reference.

Definition A (Nakano 1950) — A sequence $\{a_n\}$ in L is said to be order-convergent to $a \in L$ if there exists a sequence $\epsilon_n \overset{\infty}{\underset{n=1}{\downarrow}} 0$ in L such that

$$| a_n - a | \leqslant \epsilon_n, \; n = 1, 2, \ldots .$$

we write 0-$\lim_n a_n = a$.

Definition B (Kundu and Lahiri 1979) — A linear lattice L is said to be operative if for every set $M \subset L$ with sup M and inf M in L the following conditions hold:

(i) if $c < \sup M$, then there exists at least one $x \in M$ such that $c < x$;

(ii) if $\inf M < c$, then there exists at least one $x \in M$ such that $x < c$.

Definition C (Nakano 1950) — A linear lattice L is called sequentially continuous if for every sequence $\{a_n\}_{n=1}^{\infty}$ of elements of L, $a_n \geqslant 0$ and $\bigwedge_{n=1}^{\infty} a_n$ exists in L.

Definition D — Let L be a linear lattice, then

$$L^+ = \{a \in L \mid a \geqslant 0\}.$$

Throughout the paper X, E, F will denote linear lattices over real field.

2. Decomposition of S with respect to σ

Let $\sigma : Q(S) \to F$ be a countably additive set function with the property that $\sigma(A)$ is comparable with 0 (null element) for every $A \in Q(S)$ and that $\sigma(\phi) = 0$. We shall stick to this definition of σ throughout. In this section we take F to be sequentially continuous, complete and operative, F^+ totally ordered and E is also sequentially continuous.

With respect to σ it is easy to establish Theorems 2.1 and 2.2.

Theorem 2.1 — If $\{A_n\}$ be a non-decreasing sequence of sets in $Q(S)$, then

$$\sigma(\lim_m A_m) = 0\text{-}\lim_m \sigma(A_m)$$

provided the right-hand limit exists.

Theorem 2.2 — If $\{A_n\}$ be a non-increasing sequence of sets in $Q(S)$, then

$$\sigma(\lim_m A_m) = 0\text{-}\lim_m \sigma(A_m)$$

provided the right-hand limit exists.

Lemma 2.1 — Let F be a complete, sequentially continuous and operative linear lattice; if $A \subset F$ and $g = \sup A$, then there exists a sequence $\{a_n\}$ in A such that $0\text{-}\lim_n a_n = g$.

PROOF : Let $0 < \epsilon \in F$, since $g - (\epsilon/2^n) < g$ and F is operative, there exists $a_n \in A$ such that $g - (\epsilon/2^n) < a_n \leqslant g$, $n = 1, 2, \ldots$. Accordingly,

$$g - a_n = |g - a_n| < \epsilon/2^n,\ n = 1, 2, \ldots.$$

Since F is sequentially continuous it follows that $\epsilon/2^n \downarrow 0$ and therefore, $\{a_n\}$ order-converges to g in F.

Kundu and Lahiri (1977) proved a theorem (Theorem 3.7) in the context of a measure function ι. In fact we can prove the theorem with respect to σ as well.

Theorem 2.3 — The set function σ attains its supremum on $Q(S)$.

PROOF : Let $g = \sup \{\sigma(Y)/Y \in Q(S)\}$. In view of Lemma 2.1, there exists a sequence $\{\sigma(A_n)\}$, $n = 1, 2, \ldots$ in F such that $0\text{-}\lim_n \sigma(A_n) = g$.

If $A = \bigcup_{n=1}^{\infty} A_n$, then $A \in Q(S)$ and as such $A - A_i = K_i \in Q(S)$, $i = 1, 2, 3, \ldots$.

For a fixed positive integer m, we construct disjoint sets P_{m_j}, $j = 1, 2, \ldots, 2^m$ as follows:

$$A_1 \cap A_2 \cap \ldots \cap A_m,\ K_1 \cap A_2 \cap \ldots \cap A_m,\ A_1 \cap K_2 \cap \ldots \cap A_m, \ldots,$$
$$A_1 \cap \ldots \cap A_{m-1} \cap K_m,\ K_1 \cap K_2 \cap A_3 \cap \ldots \cap A_m, \ldots, A_1 \cap A_2 \cap \ldots$$
$$\ldots \cap K_{m-1} \cap K_m, \ldots, K_1 \cap K_2 \cap \ldots \cap K_m.$$

Clearly, $P_{m_j} \in Q(S)$, $j = 1, 2, \ldots, 2^m$.

Let $B_m = \bigcup_j P_{m_j}$, where $\sigma(P_{m_j}) \geqslant 0$. Then $B_m \in Q(S)$. Since A_m is the union of some P_{m_j}, $\sigma(B_m) \geqslant \sigma(A_m)$. For $m'' > m'$ either $P_{m''_j} \subset B_{m'}$ or, $B_{m'} \cap P_{m''_j} = \phi$. Consequently, $\sigma(B_m \cup B_{m+1} \cup \ldots \cup B_{m+k}) \geqslant \sigma(B_m) \geqslant \sigma(A_m)$. Let $D_k = \bigcup_{i=0}^{k} B_{m+i}$ then $\{D_k\}_{k=1}^{\infty}$ is monotone increasing in $Q(S)$.

$$\text{Let}\quad C_m = \lim_k D_k = \bigcup_{i=0}^{\infty} D_{m+i}.$$

Then $\{C_m\}$ forms a monotone decreasing sequence in $Q(S)$. By Theorem 2.1,

$$\sigma(C_m) = \sigma\,(\lim_k D_k) = 0\text{-}\lim_k \sigma(D_k) \geqslant \sigma(A_m).$$

Let $C = \bigcap_{m=1}^{\infty} C_m$, then $C \in Q(S)$ and $\sigma(C) \leqslant g$. ...(2.1)

By Theorem 2.2,

$$\sigma(C) = 0\text{-}\lim_m \sigma(C_m) \geqslant 0\text{-}\lim_m \sigma(A_m) = g. \qquad ...(2.2)$$

From (2.1) and (2.2) we obtain $\sigma(C) = g$.

Hence the theorem.

Definition 2.1 (Kundu and Lahiri 1977) — For $A \in Q(S)$, we define

$$Q_A = \{B/B \subseteq A,\ B \in Q(S)\}.$$

Definition 2.2 (Kundu and Lahiri 1977) — For $A \in Q(S)$, we define

$$\sigma_+(A) = \sup\,\{\sigma(B)/B \in Q_A\}$$
$$\sigma_-(A) = -\inf\,\{\sigma(B)/B \in Q_A\}.$$

Since $\phi \subseteq A$ and $\sigma(\phi) = 0$ we have $\sigma_+(A) \geqslant 0$ and $\sigma_-(A) \geqslant 0$ for every $A \in Q(S)$.

Theorem 2.4 — σ_+ and σ_- have isotone property, and $-\sigma_-(A) \leqslant \sigma(A) \leqslant \sigma_+(A)$ for every $A \in Q(S)$.

PROOF: Proof is obvious.

Lemma 2.2 — For every $A \in Q(S)$, $\sigma(A) = \sigma_+(A) - \sigma_-(A)$.

PROOF: In view of Lemma 2.1 and Definition 2.2, we obtain a sequence $\{\sigma(A_n)\}$ in F such that $0 - \lim_n \sigma(A_n) = \sigma_+(A)$, where $A_n \in Q_A$, $n = 1, 2 \ldots$. Since for each n, $\sigma(A) = \sigma(A_n) + \sigma(A - A_n)$, it follows that

$$\begin{aligned} 0 - \lim_n \sigma(A - A_n) &= \sigma(A) - 0\text{-}\lim_n \sigma(A_n) \\ &= \sigma(A) - \sigma_+(A) \\ &= \sigma(A) - \sup\,\{\sigma(Y)/Y \in Q_A\} \\ &= \sigma(A) + \inf\,\{-\,\sigma(Y)/Y \in Q_A\} \\ &= \inf\,\{(\sigma(A) - \sigma(Y))/Y \in Q_A\} \\ &= \inf\,\{\sigma(A - Y)/Y \in Q_A\} \\ &= -\sigma_-(A). \end{aligned}$$

Therefore, we get $\sigma(A) = \sigma_+(A) - \sigma_-(A)$.

Theorem 2.5 — For every $A \in Q(S)$, (i) if $\sigma_+(A) = 0$, then $\sigma(A) \leqslant 0$ and $\sigma(A \cap B) \leqslant 0$ for every $B \in Q(S)$, and (ii) if $\sigma_-(A) = 0$, then $\sigma(A) \geqslant 0$ and $\sigma(A \cap B) \geqslant 0$ for every $B \in Q(S)$.

PROOF : Let $\sigma_+(A) = 0$, $A \in Q(S)$; then in view of Lemma 2.2,

$$\sigma(A) = \sigma_+(A) - \sigma_-(A) \leqslant 0.$$

Since $A \cap B \subseteq A$ for every $B \in Q(S)$ and σ_+ is isotone,

$$0 \leqslant \sigma_+(A \cap B) \leqslant \sigma_+(A) = 0$$

giving $\sigma_+(A \cap B) = 0$ and consequently by theorem 2.4, $\sigma(A \cap B) \leqslant \sigma_+(A \cap B) = 0$ for every $B \in Q(S)$. Hence the case (i).

We can prove (ii) similarly.

Theorem 2.6 — For every $A \in Q(S)$, there is a decomposition $A = A^+ \cup A^-$ such that $A^+ \cap A^- = \phi$ and $\sigma_+(A^-) = 0 = \sigma_-(A^+)$.

PROOF : By Theorem 2.3, there exists $C \in Q(S)$ such that

$$\sigma(C) = \sup \{\sigma(Y)/Y \in Q(S)\}.$$

Let $A^+ = A \cap C$ and $A^- = A - A^+$; clearly $A^+ \cap A^- = \phi$. We claim that $\sigma_+(A^-) = 0$. For if $\sigma_+(A^-) \neq 0$ then $\sigma_+(A^-) > 0$ and hence there exists at least one subset $D \subset A^-$, $D \in Q(S)$ such that $\sigma(D) > 0$.

$$\sigma(D \cup C) = \sigma(D) + \sigma(C) > \sigma(C).$$

Since $D \cup C \in Q(S)$, this contradicts the fact that σ attains supremum at C.

Next we claim that $\sigma_-(A^+) = 0$; otherwise $\sigma_-(A^+) > 0$ i.e. $-\sigma_-(A^+) < 0$. Then there is one $D \subset A^+$ i.e. $D \subset C$ such that $\sigma(D) < 0$. Now

$$\sigma(C - D) = \sigma(C) - \sigma(D) > \sigma(C),$$

since $C - D \in Q(S)$, it is a contradiction.

Hence the theorem.

Theorem 2.7 — If $\nu : Q(S) \to F$ be a measure and $\epsilon(> 0) \in E$, then the set function $\sigma = \nu - \epsilon\iota$ is countably additive, $\sigma(A)$ is comparable with 0 for every $A \in Q(S)$ and $\sigma(\varphi) = 0$.

PROOF : Proof is obvious.

Definition 2.3 — Let $\nu : Q(S) \to F$ be a measure. Then ν is said to be absolutely continuous with respect to the measure ι, in symbol $\nu \lll \iota$ if $\nu(A) = 0$ for every $A \in Q(S)$ whenever $\iota(A) = 0$.

Theorem 2.8 — If $\nu : Q(S) \to F$ be a measure not identically zero such that $\nu \ll \iota$, then there exists $\epsilon\ (> 0) \in E$ and a set $A \in Q(S)$ such that $\iota(A) > 0$ and such that for the set function $\nu - \epsilon\iota$, $(\nu - \epsilon\iota)(A \cap B) \geqslant 0$ for every $B \in Q(S)$.

PROOF : In view of Theorem 2.7, we have in accordance with Theorem 2.6, $S = A_n \cup B_n$, $A_n \cap B_n = \phi$ as a decomposition of S with respect to the set function $\sigma_n = \nu - \frac{\epsilon'}{n}\iota$, $\epsilon' \in E$ such that $(\sigma_n)_+ (B_n) = 0$ and $(\sigma_n)_- (A_n) = 0$, $n = 1, 2, \ldots$.

Let $A_0 = \bigcup_{n=1}^{\infty} A_n$ and $B_0 = \bigcap_{n=1}^{\infty} B_n$. Then $A_0, B_0 \in Q(S)$.

$$S - A_0 = A_0^C = (\bigcup_{n=1}^{\infty} A_n)^C = \bigcap_{n=1}^{\infty} A_n^C = \bigcap_{n=1}^{\infty} B_n = B_0$$

Since $B_0 \subset B_n$, $(\sigma_n)_+$ is isotone and $(\sigma_n)_+ (B_n) = 0$, we have $(\sigma_n)_+ (B_0) = 0$. Consequently, by Theorem 2.5, $(\sigma_n)(B_0) \leqslant 0$. Hence $\nu(B_0) \leqslant \frac{\epsilon'}{n} \iota(B_0)$, $n = 1, 2, \ldots$.

Since E is sequentially continuous, we have $\nu(B_0) = 0$ (Kundu and Lahiri 1979), giving $\nu(S - A_0) = 0$.

Consequently, $\nu(A_0) = \nu(S) > 0$.

But since $\nu \ll \iota$ and $\nu(A_0) > 0$ it follows that $\iota(A_0) > 0$.

$$0 < \iota(A_0) = \iota(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \iota(A_n)$$

and hence $\iota(A_n) > 0$ for at least one n. For such a 'n', we write $A = A_n$ and $\epsilon'/n = \epsilon$, the requirements of the theorem are all satisfied in view of Theorem 2.5.

3. RADON-NIKODYM THEOREM

In this section we assume the following restrictions on X, E, F : X, E, F are complete of which F is operative and sequentially continuous, E is operative and X^+, E^+, F^+ are linearly ordered.

Before we establish the main theorem, we mention some known definitions and results. Our discussions in this section will be in the context of the measure space $(S, Q(S), \iota, X)$.

Let $\mathcal{E}_E(Q(S))$ be the class of all E-valued simple functions (Kundu and Lahiri 1979); if $f(x) = \sum_{i=1}^{n} c_i \chi_{B_i}(x)$ be the canonical representation of a simple function $f \in \mathcal{E}_E(Q(S))$, then for any $B \in Q(S)$ we define $\phi_B : \mathcal{E}_E(Q(S)) \to F$ by the rule

$$\phi_B(f) = \sum_{i=1}^{n} \iota(B \cap B_i)\, c_i.$$

We write ϕ instead of ϕ_S.

Definition 3.1 (Kundu and Lahiri 1979) — A sequence $\{f_n\}$ in $\mathcal{E}_E(Q(S))$ is called a Cauchy sequence iff $0\text{-}\lim_{m,n} \phi(\,|f_n(x) - f_m(x)|\,) = 0$ for every $x \in S$.

Definition 3.2 (Kundu and Lahiri 1979) — A function $f : S \to E \cup \{\infty\}$ is said to be integrable (or ι-integrable) on a measurable set $B \in Q(S)$, if there exists a Cauchy sequence $\{f_n\}$ in $\mathcal{E}_E(Q(S))$ order converging a.e. to f such that $0\text{-}\lim_n \phi_B(f_n) < \infty$. In this case we write $\int_B f\, d\iota = 0\text{-}\lim_n \phi_B(f_n)$. For $\int_S f\, d\iota$ we write $\int f\, d\iota$. It may be shown that f be integrable over S then it is also integrable over any measurable subset A of S.

Let $L_E(S, \iota)$ be the collection of all ι-integrable functions $f : S \to E \cup \{\infty\}$. For brevity we shall denote $L_E(S, \iota)$ by $L_E(\iota)$. It can be shown that $\mathcal{E}_E(Q(S)) \subset L_E(\iota)$.

We list a number of properties of $L_E(\iota)$ in the form of a theorem.

Theorem 3.1 — $L_E(\iota)$ is a linear lattice over reals and if $f, g \in L_E(\iota)$ then for every pair of scalar α, β and $B, B_i \in Q(S)$

(i) $\int_B (\alpha f + \beta g)\, d\iota = \alpha \int_B f\, d\iota + \beta \int_B g\, d\iota$;

(ii) $\int_{\bigcup_{i=1}^{n} B_i} f\, d\iota = \sum_{i=1}^{n} \int_{B_i} f\, d\iota$, $B_i \cap B_j = \phi$, $i \neq j$;

(iii) $\left| \int_B f\, d\iota \right| \leqslant \int_B |f|\, d\iota$.

Proof is omitted.

Definition 3.3 — The indefinite integral of a function $f(x) \in L_E(\iota)$ is the set function $\nu : Q(S) \to F$ defined by $\nu(A) = \int_A f\, d\iota$ for every $A \in Q(S)$.

Theorems 3.2 and 3.3 are consequences of Theorem 3.1.

Theorem 3.2 — The indefinite integral of an integrable function is finitely additive.

Theorem 3.3 — If an integrable function $f(x) \geqslant 0$ a.e., then its indefinite integral is isotone.

Theorem 3.4 — The indefinite integral of an integrable function is countably additive.

PROOF: Let $A_i \in Q(S)$, $i = 1, 2, \ldots$; $A_i \cap A_j = \phi$, $i \neq j$ and $A = \bigcup_{i=1}^{\infty} A_i$.

Case I: $f(x) \in \mathcal{E}_E((QS))$

Let $f(x)$ have the canonical representation $f(x) = \sum_{j=1}^{n} c_j \chi_{B_j}(x)$. Then

$$A_i \cap B_j \in Q(S)$$

for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots n$.

Hence
$$\int_A f\, d\iota = \phi_A(f) = \sum_{j=1}^{n} c_j\, \iota(A \cap B_j)$$

$$= \sum_{j=1}^{n} c_j\, \iota\, [(\bigcup_{i=1}^{\infty} A_i) \cap B_j]$$

$$= \sum_{j=1}^{n} c_j \sum_{i=1}^{\infty} \iota(A_i \cap B_j)$$

$$= \sum_{j=1}^{n} [0\text{-}\lim_k \sum_{i=1}^{k} c_j\, \iota(A_i \cap B_j)]$$

$$= 0\text{-}\lim_k \sum_{i=1}^{k} [\sum_{j=1}^{n} c_j\, \iota(A_i \cap B_j)]$$

$$= 0\text{-}\lim_k \sum_{i=1}^{k} \phi_{A_i}(f)$$

$$= \sum_{i=1}^{\infty} \phi_{A_i}(f) = \sum_{i=1}^{\infty} \int_{A_i} f\, d\iota.$$

Case II: $f(x) \in L_E(\iota)$

Since $f(x) \in L_E(\iota)$ there exists a Cauchy sequence $\{f_n\}$ in $\mathcal{E}_E(Q(S))$ order converging a.e. to $f(x)$ such that

$$\nu(A) = \int_A f(x)\, d\iota = 0\text{-}\lim_n \int_A f_n(x)\, d\iota < \infty.$$

From the definition of order limit there exists a sequence $\omega_n \downarrow 0$ in F such that

$$|\, \nu(A) - \int_A f_n\, d\iota\, | \leqslant \omega_n,\ n = 1, 2, \ldots . \qquad \ldots(3.1)$$

By Theorem 3.2,

$$\nu\Big(\bigcup_{i=1}^{k} A_i\Big) = \int_{\bigcup_{i=1}^{k} A_i} f\,d\iota = \sum_{i=1}^{k} \int_{A_i} f\,d\iota$$

$$= \sum_{i=1}^{k} 0\text{-}\lim_n \int_{A_i} f_n d\iota$$

$$= 0\text{-}\lim_n \Big(\sum_{i=1}^{k} \int_{A_i} f_n\,d\iota\Big).$$

Hence, there exists a sequence $\omega_n' \downarrow 0$ in F such that

$$\Big| \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota - \nu\Big(\bigcup_{i=1}^{k} A_i\Big) \Big| \leqslant \omega_n', \; n = 1, 2, \ldots . \qquad \ldots(3.2)$$

Also by Case I, $\int_A f_n\,d\iota = \sum_{i=1}^{\infty} \int_{A_i} f_n\,d\iota = 0\text{-}\lim_k \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota.$

Therefore there exists a sequence $\omega_n'' \downarrow 0$ in F such that

$$\Big| \int_A f_n\,d\iota - \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota \Big| \leqslant \omega_k'', \; k = 1, 2, \ldots . \qquad \ldots(3.3)$$

Then $$\Big| \nu(A) - \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota \Big|$$

$$= \Big| \nu(A) - \int_A f_n\,d\iota + \int_A f_n\,d\iota - \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota + \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota - \sum_{i=1}^{k} \nu(A_i) \Big|$$

$$\leqslant \Big| \nu(A) - \int_A f_n\,d\iota \Big| + \Big| \int_A f_n\,d\iota - \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota \Big| + \Big| \sum_{i=1}^{k} \int_{A_i} f_n\,d\iota - \sum_{i=1}^{k} \nu(A_i) \Big|$$

$$\leqslant \omega_n + \omega_k'' + \omega_n', \text{ by (3.1), (3.2) and (3.3).}$$

Since n is independent of k we take $n = k$ and write $\omega_k''' = \omega_k + \omega_k' + \omega_k''$. As $\omega_k''' \downarrow 0$, and obtain $\nu(A) = \sum_{i=1}^{\infty} \nu(A_i)$.

Thus the indefinite integral of a function $f(x) \in L_E(\iota)$ such that $f(x) \geqslant 0$ a.e. has the properties of measure and hence we consider it as a measure function.

Theorem 3.5 — Let $\nu : Q(S) \to F$ be a measure such that $\nu \ll \iota$, then there is a function $f \in L_E(\iota)$ such that

$$\nu(A) = \int_A f \, d\iota, \text{ for all } A \in Q(S).$$

PROOF: Let $\mathcal{K}$ be the class of all non-negative functions $f \in L_E(\iota)$ such that $\int_A f \, d\iota \leqslant \nu(A)$ for every $A \in Q(S)$.

We write $\alpha = \sup_{f \in \mathcal{K}} \{\int f \, d\iota\}$.

Since F is operative there exists a sequence of functions $\{f_n\}$ in $\mathcal{K}$ such that

$$0\text{-}\lim_n \int f_n d\iota = \alpha. \qquad \ldots(3.4)$$

For any fixed set $A \in Q(S)$ and for fixed integer n, let

$$g_n(x) = f_1(x) \vee f_2(x) \vee \ldots \vee f_n(x); f_1, f_2, \ldots, f_n \in \mathcal{K}$$

and $A_i = \{x \in A \mid f_i(x) = g_n(x)\}$.

Clearly, the sets A_i, $i = 1, 2, \ldots, n$ are disjoint and $A = \bigcup_{i=1}^{n} A_i$. Also $g_n \in L_E(\iota)$ by Theorem 3.1.

Consequently, we have

$$\int_A g_n \, d\iota = \sum_{i=1}^{n} \int_{A_i} f_i \, d\iota \leqslant \sum_{i=1}^{n} \nu(A_i) = \nu(A) \qquad \ldots(3.5)$$

which implies that $g_n \in \mathcal{K}$ for $n = 1, 2, \ldots$. Let $f_0(x) = \sup_n \{f_n(x)\} = \sup_n \{g_n(x)\}$. Since F is complete, $\sup_n \int g_n \, d\iota$ exists in F and since $\{g_n(x)\}$ is a non-decreasing sequence of functions in $L_E(\iota)$ by Theorem 5.5 (Kundu and Lahiri 1979) we have

$$0\text{-}\lim_n \int g_n \, d\iota = \sup_n \int g_n \, d\iota = \int \sup_n g_n \, d\iota = \int f_0 \, d\iota \leqslant \alpha.$$

Also from (3.4), $\alpha = 0 - \lim_n \int f_n \, d\iota \leqslant 0 - \lim_n \int g_n \, d\iota$ so that we obtain

$$\alpha = \int f_0 \, d\iota. \qquad \ldots(3.6)$$

So, f_0 is ι-integrable on S. Hence it is integrable on any $A \in Q(S)$. Now it is easy to see by Theorem 5.5 (Kundu and Lahiri 1979) that for any measurable set A, $\int_A f_0 \, d\iota = \sup_n \int_A g_n \, d\iota \leqslant \nu(A)$ (by (3.5)), so that $f_0 \in \mathcal{K}$.

Let us define a set function ν_0 on $Q(S)$ by

$$\nu_0(A) = \nu(A) - \int_A f_0 \, d\iota.$$

Clearly $\nu_0(A) \geqslant 0$ for all $A \in Q(S)$. Also, it follows immediately from Theorem 3.4 that ν_0 is countably additive set function absolutely continuous with respect to ι and $\nu_0(\phi) = 0$.

Let $A, B \in Q(S)$ and $A \subset B$, then

$$\nu_0(B) = \nu_0(A) + \nu_0(B \setminus A),$$

i.e.

$$\nu_0(B) - \nu_0(A) = \nu_0(B \setminus A)$$

$$= \nu(B \setminus A) - \int_{B \setminus A} f_0 \, d\iota \geqslant 0$$

giving that ν_0 has the isotone property. Therefore, $\nu_0 : Q(S) \to F$ is a measure.

We shall now show that ν_0 is identically zero. If ν_0 is not identically zero then it follows from Theorem 2.8 that there exists $\epsilon(>0)$ in E and a set $A \in Q(S)$ such that $\iota(A) > 0$ and such that for the set function $\sigma = \nu_0 - \epsilon\iota$, $\sigma(A \cap B) \geqslant 0$ for every $B \in Q(S)$.

Hence $\nu_0(A \cap B) \geqslant \epsilon\iota(A \cap B)$ for every $B \in Q(S)$ which implies that

$$\epsilon\iota(A \cap B) \leqslant \nu(A \cap B) - \int_{A \cap B} f_0 \, d\iota \qquad \ldots(3.7)$$

We construct a function $g = f_0 + \epsilon\chi_A$, then

$$\int_B g \, d\iota = \int_B f_0 \, d\iota + \int_B \epsilon\chi_A \, d\iota$$

$$= \int_{B-A} f_0 \, d\iota + \int_{B \cap A} f_0 \, d\iota + \epsilon\iota(B \cap A)$$

$$\leqslant \int_{B-A} f_0 \, d\iota + \int_{B \cap A} f_0 \, d\iota + \nu(B \cap A) - \int_{B \cap A} f_0 \, d\iota, \text{ by (3.7)}$$

$$\leqslant \nu(B - A) + \nu(B \cap A) = \nu(B), \text{ for every } B \in Q(S).$$

Therefore, $g \in \mathcal{K}$.

Also,

$$\int g \, d\iota = \int f_0 \, d\iota + \int \epsilon\chi_A \, d\iota$$

$$= \int f_0 \, d\iota + \epsilon\iota(A)$$

$$> \int f_0 \, d\iota = \alpha, \text{ by (3.6)}$$

This contradicts the maximality of $\int f_0 \, d\iota$.

Hene $\nu_0(A) = 0$ for every $A \in Q(S)$. If we take $f_0 = f$, then $\nu(A) = \int_A f \, d\iota$ for all $A \in Q(S)$.

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