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ON IDENTITIES OF ROGERS-RAMANUJAN TYPE

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By using Bailey's transformation identities of Rogers-Ramanujan type with a positive integer as a free parameter are obtained. These identities besides generalizing the known identities of Rogers-Ramanujan type, also unify, to a great extent, identities on the same modulus.

§1. Starting with the simple identity

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \qquad \ldots(1.1)$$

where $\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q]_{n-r}\,[aq]_{n+r}}$ and $\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{[q]_{r-n}\,[aq]_{r+n}}$, Bailey (1947) showed that by choosing $\delta_r = [\rho_1]_r\,[\rho_2]_r \left(\frac{aq}{\rho_1\rho_2}\right)^r$, the sequence $\langle \gamma_n \rangle$ can be evaluated by the q-analogue of Gauss' summation theorem:

$${}_2\phi_1\left[\begin{matrix} a, b; c/ab \\ c \end{matrix}\right] = \frac{[c/a]_\infty\,[c/b]_\infty}{[c]_\infty\,[c/ab]_\infty},\ |\ c/ab\ | < 1 \qquad \ldots(1.2)$$

(hence onwards we stick to the convention that in all the basic hypergeometric series as well as in the products the base is q, $|\ q\ | < 1$ unless mentioned otherwise) so that (1.1) yields the identity

$$\sum_{n=0}^{\infty} [\rho_1]_n\,[\rho_2]_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n = \frac{[aq/\rho_1]_\infty\,[aq/\rho_2]_\infty}{[aq]_\infty\,[aq/\rho_1\rho_2]_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{[\rho_1]_n\,[\rho_2]_n}{[aq/\rho_1]_n\,[aq/\rho_2]_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n \qquad \ldots(1.3)$$

where $\qquad \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q]_{n-r}\,[aq]_{n+r}}.$

From (1.3) Bailey (1947) obtained Rogers-Ramanujan type of identities by choosing the sequence $\langle \alpha_n \rangle$ suitably. It was pointed out by Bailey (1949) that if the following summation formula is used

$$ {}_2\phi_1\begin{bmatrix} a, b; c/ab \\ cq \end{bmatrix} = \frac{[cq/a]_\infty [cq/b]_\infty}{[cq]_\infty [cq/ab]_\infty} \left\{\frac{ab(1+c) - c(a+b)}{ab - c}\right\} \quad \ldots(1.4) $$

instead of using (1.2), we get the identity

$$ \sum_{n=0}^{\infty} [\rho_1]_n [\rho_2]_n \left(\frac{a}{\rho_1\rho_2}\right)^n \beta_n = \frac{[aq/\rho_1]_\infty [aq/\rho_2]_\infty}{[aq]_\infty [aq/\rho_1\rho_2]_\infty} $$

$$ \times \sum_{n=0}^{\infty} \frac{[\rho_1]_n [\rho_2]_n}{[aq/\rho_1]_n [aq/\rho_2]_n} \left(\frac{a}{\rho_1\rho_2}\right)^n $$

$$ \times \left[\frac{\rho_1\rho_2(1 + aq^{2n}) - aq^n(\rho_1 + \rho_2)}{\rho_1\rho_2 - a}\right] \alpha_n \quad \ldots(1.5) $$

from which identities of Rogers-Ramanujan type were deduced. Slater (1951, 1952) used the identity (1.3) of Bailey and gave a long list of Rogers-Ramanujan type of identities.

It is, therefore, natural to expect that if one uses the following transformation (Verma 1977): If either a or b or c is of the form q^{-m-1}, m an integer $\geqslant -1$, then

$$ {}_2\phi_1\begin{bmatrix} a, b; ec/ab \\ e \end{bmatrix} = \frac{[e/a]_\infty [e/b]_\infty}{[e]_\infty [e/ab]_\infty} \, {}_3\phi_1\begin{bmatrix} a, b, c; q \\ abq/e \end{bmatrix}, \left|\frac{ec}{ab}\right| < 1 \quad \ldots(1.6) $$

(If a or b is of the form q^{-m-1} the basic hypergeometric series on both the sides terminate and hence no convergence condition is required. On the other hand if neither a nor b is of the form q^{-m-1}, whereas $c = q^{-m-1}$, then for the convergence of the basic hypergeometric series on the left-hand side we require $| ec/ab | < 1$.) and by using (1.6) for evaluating $\langle \gamma_n \rangle$ instead of using (1.2) or (1.4) [because (1.6) is a generalization of (1.2) and (1.4) and reduces to (1.2) for $m = -1$ and to (1.4) for $m = 0$] identities of Rogers-Ramanujan type would be obtained which would incorporate the identities obtained from (1.3) and (1.5) as special cases. In fact on making use of (1.6) for evaluating the sequence $\langle \gamma_n \rangle$ we obtain Rogers-Ramanujan type of identities with an integer as free parameter provided we choose $\langle \alpha_n \rangle$ suitably so that $\langle \beta_n \rangle$ can be evaluated in a closed form. To illustrate our point of view we have chosen in §2 the following six sets of sequences $\langle \alpha_n \rangle$ for which $\langle \beta_n \rangle$ are known in closed form:

(i) If $\alpha_0 = 1$, $\alpha_n = (-)^n (1 - aq^{2n}) \dfrac{[aq]_{n-1}}{[q]_n} a^n q^{n(3n-1)/2}$ for $n \geqslant 1$,

then $\beta_n = \dfrac{1}{[q]_n}$;

(ii) If $\alpha_0 = 1$, $\alpha_{3n} = (-)^n \dfrac{[aq^3]_{n-1,q^3} (1 - aq^{6n})}{[q^3]_{n,q^3}} a^n q^{3n(3n-1)/2}$, $\alpha_{3n-1} = \alpha_{3n-2} = 0$, for $n \geqslant 1$, then $\beta_0 = 1$ and $\beta_n = \dfrac{[aq^3]_{n-1,q^3}}{[q]_n\,[aq]_{2n-1}}$ for $n \geqslant 1$;

(iii) If $\alpha_0 = 1$, $\alpha_n = (1 - aq^{2n}) \dfrac{[aq]_{n-1}\,[b]_n}{[q]_n\,[aq/b]_n} \dfrac{a^n q^{n^2}}{b^n}$ for $n \geqslant 1$,

then $\beta_n = \dfrac{[-(a/b)\,q^{n+1}]_n}{[q^2]_{n,q^2}\,[-aq]_{2n}\,[aq/b]_n}$;

(iv) If $\alpha_0 = 1$, $\alpha_n = (-)^n \dfrac{(1 - aq^{2n})\,[aq]_{n-1}}{[q]_n}\, a^n q^{n(3n-1)/2}$ for $n \geqslant 1$,

then $\beta_n = \dfrac{[aq]_{3n}}{[q^3]_{n,q^3}\,[a^3q^3]_{2n,q^3}}$;

(v) If $\alpha_0 = 1$, $\alpha_{2n} = \dfrac{[aq^2]_{n-1,q^2}\,[f]_{n,q^2}\,(1 - aq^{4n})}{[q^2]_n\,[(a/f)\,q^2]_{n,q^2}} \left(\dfrac{a}{f}\right)^n q^{2n^2}$, $\alpha_{2n-1} = 0$

for $n \geqslant 1$, then $\beta_n = \dfrac{[aq/f]_{n,q^2}}{[q]_{n,q^2}\,[aq]_{n,q^2}\,[aq/f]_n}$;

(vi) If $\alpha_0 = 1$, $\alpha_{3n-1} = -q^{6n^2-5n+1}$, $\alpha_{3n} = q^{6n^2-n} + q^{6n^2+n}$,

$\alpha_{3n-2} = -q^{6n^2-7n+2}$ for $n \geqslant 1$, then $\beta_n = \dfrac{1}{[q]_{2n}}$

[(i) - (v) are due to Bailey (1949) whereas (vi) is due to Slater 1952, A(1)].

The advantage of the Rogers-Ramanujan type of identities proved herein, besides their generality, lies in the fact that a number of identities on a particular modulus which have hitherto appeared as scattered and isolated results can now be seen to be special cases of a very few identities on the same modulus.

§2. In (1.1) if we take $\delta_r = [\rho_1]_r\,[\rho_2]_r \left(\dfrac{a}{\rho_1\rho_2 q^m}\right)^r$ and use (1.6) for evaluating the sequence $\langle \gamma_n \rangle$ we get the following identity incorporating both (1.3) and (1.5)

$$\sum_{n=0}^{\infty} [\rho_1]_n\,[\rho_2]_n \left(\frac{a}{\rho_1\rho_2 q^m}\right)^n \beta_n = \frac{[aq/\rho_1]_\infty\,[aq/\rho_2]_\infty}{[aq]_\infty\,[aq/\rho_1\rho_2]_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{[\rho_1]_n\,[\rho_2]_n}{[aq/\rho_1]_n\,[aq/\rho_2]_n} \left(\frac{a}{\rho_1\rho_2 q^m}\right)^n {}_3\phi_1\left[\begin{matrix}\rho_1 q^n,\ \rho_2 q^n,\ q^{-m-1};\ q\\ \rho_1\rho_2/a\end{matrix}\right] \alpha_n. \quad \text{...(2.1)}$$

Now choosing our sequence $\langle \alpha_n \rangle$ to be (i), we obtain

$$\sum_{n=0}^{\infty} \frac{[\rho_1]_n\,[\rho_2]_n}{[q]_n} \left(\frac{a}{\rho_1\rho_2 q^m}\right)^n = \frac{[aq/\rho_1]_\infty\,[aq/\rho_2]_\infty}{[aq]_\infty\,[aq/\rho_1\rho_2]_\infty} \left\{ {}_3\phi_1\left[\begin{matrix}\rho_1,\ \rho_2,\ q^{-m-1};\ q\\ \rho_1\rho_2/a\end{matrix}\right] + \right.$$

(*equation continued on p.* 773)

$$+ \sum_{n=1}^{\infty} (-)^n (1 - aq^{2n}) \frac{[aq]_{n-1} [\rho_1]_n [\rho_2]_n}{[q]_n [aq/\rho_1]_n [aq/\rho_2]_n} \left(\frac{a^2}{\rho_1\rho_2}\right)^n q^{n(3n-2m-1)/2}$$

$$\times {}_3\phi_1 \left[\begin{matrix} \rho_1 q^n, \rho_2 q^n, q^{-m-1}; q \\ \rho_1\rho_2/a \end{matrix}\right]\Bigg\}. \qquad ...(2.2)$$

Letting $\rho_1, \rho_2 \to \infty$ in (2.2), we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)-mn}}{[q]_n} a^n = \frac{1}{[aq]_\infty} \Bigg[\sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} a^s q^{s(s+1)/2}$$

$$+ \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} a^s q^{s(s+1)/2} \sum_{n=1}^{\infty} (-)^n (1 - aq^{2n})$$

$$\times \frac{[aq]_{n-1}}{[q]_n} a^{2n} q^{(n(5n-3)/2)-mn+2ns}\Bigg]. \qquad ...(2.3)$$

For $a = 1$, (2.3) yields

$$[q]_\infty \sum_{n=0}^{\infty} \frac{q^{n(n-1)-mn}}{[q]_n} = \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^{s(s+1)/2}$$

$$\times \prod_{n=1}^{\infty} \Big\{(1 - q^{5n-4-m+2s}) (1 - q^{5n-1+m-2s}) (1 - q^{5n})\Big\}. \qquad ...(2.4)$$

Setting $m = -1$ in (2.4) we have one of the famous Rogers-Ramanujan identities [Slater 1952, eqn. (18)]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{[q]_n} = \prod_{n \not\equiv 0, 2, 3 \pmod 5} (1 - q^n)^{-1}. \qquad ...(2.5)$$

Setting $m = 0$ in (2.4) and making use of (2.5), we get the other famous Rogers-Ramanujan identity [Slater 1952, eqn. (14)]

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{[q]_n} = \prod_{n \not\equiv 0, 1, 4 \pmod 5} (1 - q^n)^{-1}. \qquad ...(2.6)$$

On the other hand for $a = q$, (2.3) yields on some simplification

$$[q]_\infty \sum_{n=0}^{\infty} \frac{q^{n^2-nm}}{[q]_n} = \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^{s(s+3)/2} \times$$

(*equation continued on p. 774*)

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{5n-2-m+2s})(1 - q^{5n-3+m-2s})(1 - q^{5n})\right\}. \qquad \ldots(2.7)$$

Now comparing (2.4) and (2.7), we get the following equivalent product relation

$$\sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^{s(s+1)/2} \prod_{n=1}^{\infty} \left\{(1 - q^{5n-4-m+2s})(1 - q^{5n-1+m-2s})\right\}$$

$$= \sum_{s=0}^{m+2} (-)^s \frac{[q^{-m-2}]_s}{[q]_s} q^{s(s+3)/2} \prod_{n=1}^{\infty} \left\{(1 - q^{5n-3-m+2s})(1 - q^{5n-2+m-2s})\right\}. \qquad \ldots(2.8)$$

Furthermore, in (2.2) letting $\rho_2 \to \infty$, $\rho_1 = -\sqrt{aq}$, we have

$$\sum_{n=0}^{\infty} \frac{[-\sqrt{aq}]_n}{[q]_n} a^{n/2} q^{n(n-2-2m)/2} = \frac{[-\sqrt{aq}]_\infty}{[aq]_\infty} \left\{ {}_2\phi_0 \left[\begin{matrix} q^{-m-1}, & -\sqrt{aq}; \sqrt{aq} \\ - & \end{matrix} \right] \right.$$

$$+ \sum_{n=1}^{\infty} (-)^n \frac{[aq]_{n-1}}{[q]_n} (1 - aq^{2n}) a^{(3/2)n} q^{n(4n-3-2m)/2}$$

$$\left. \times {}_2\phi_0 [q^{-m-1}, -q^n \sqrt{aq}; -; -q^n \sqrt{aq}] \right\}. \qquad \ldots(2.9)$$

For $a = q$, (2.9) gives

$$\sum_{n=0}^{\infty} \frac{[-q]_n}{[q]_n} q^{n(n-1-2m)/2} = \frac{[-q]_\infty}{[q]_\infty} \sum_{n=0}^{\infty} (-)^n (1 - q^{2n+1}) q^{n(2n-m)}$$

$$\times \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} [-q^{n+1}]_s q^{s(n+1)}$$

$$= \frac{[-q]_\infty}{[q]_\infty} \sum_{n=0}^{\infty} (-)^n (1 - q^{2n+1}) q^{n(2n-m)} \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^{s(n+1)}$$

$$\times \sum_{j=0}^{s} (-)^j \frac{[q^{-s}]_j}{[q]_j} q^{j(s+n+1)}$$

(*equation continued on p.* 775)

$$= \frac{[-q]_\infty}{[q]_\infty} \left\{ \sum_{s=0}^{m+1} \sum_{j=0}^{s} (-)^{s+j} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{s+j+sj} \sum_{n=0}^{\infty} (-)^n q^{n(2n-m+s+j)} \right.$$

$$\left. + \sum_{s=0}^{m+1} \sum_{j=0}^{s} (-)^{s+j} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{m+1+sj} \sum_{n=-\infty}^{-1} (-)^n q^{n(2n+m-s-j)} \right\}$$

$$= \frac{[-q]_\infty}{[q]_\infty} \left\{ \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{s+j+sj} \sum_{n=0}^{\infty} (-)^n q^{n(2n-m+s+j)} \right.$$

$$\left. + \sum_{r=0}^{m+1} \sum_{l=r}^{m+1} (-)^{r+l} \frac{[q^{-m-1}]_l [q^{-l}]_r}{[q]_r [q]_l} q^{r+l+rl} \sum_{n=-\infty}^{-1} (-)^n q^{n(2n-m+r+l)} \right\}$$

(the second summand of the last line is obtained from the second summand of the previous line by first setting $s = m + 1 - r$ and then $j = m + 1 - l$)

$$= \frac{[-q]_\infty}{[q]_\infty} \sum_{s=0}^{m+1} \sum_{j=s}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{s+j+sj}$$

$$\times \prod_{n=1}^{\infty} \left\{ (1 - q^{4n+j+s-m-2})(1 - q^{4n-j-s+m-2})(1 - q^{4n}) \right\}. \quad \text{...(2.10)}$$

For $m = -1$, (2.10) reduces to

$$\sum_{n=0}^{\infty} \frac{[-q]_n}{[q]_n} q^{n(n+1)/2} = [-q]_\infty \prod_{n=0}^{\infty} (1 - q^{4n+2})^{-1}. \quad \text{...(2.11)}$$

For $m = 0$, (2.10) yields (on making use of (2.11))

$$\sum_{n=0}^{\infty} \frac{[-q^2]_n}{[q]_n} q^{n(n+1)/2} = [-q^2]_\infty \prod_{n=0}^{\infty} \left[\frac{(1 - q^{4n+2})^2}{(1 - q^{4n+1})(1 - q^{4n+3})} \right]. \quad \text{...(2.12)}$$

However, in (2.9) setting $a = 1$ and proceeding exactly as above, we get a result analogous to (2.10) viz.

$$\sum_{n=0}^{\infty} \frac{[-q^{1/2}]_n}{[q]_n} q^{n(n-2m-2)/2} = \frac{[-\sqrt{q}]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_j [q^{-j}]_s}{[q]_s [q]_j}$$

$$\times q^{(s+j+2sj)/2} \prod_{n=1}^{\infty} \left\{ (1 - q^{4n-m+s+j-(7/2)})(1 - q^{4n+m-s-j-(1/2)})(1 - q^{4n}) \right\}.$$

...(2.13)

Further, starting from (2.2) by letting $\rho_2 \to \infty$, $\rho_1 = \sqrt{a}$ and then setting $a = 1$, we get the following identity of Rogers-Ramanujan type:

$$\sum_{n=0}^{\infty} \frac{[-1]_n}{[q]_n} q^{n(n-2m-1)/2} = \frac{[-q]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{s(1+j)}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{4n+s+j-m-3})(1 - q^{4n-s-j+m-1})(1 - q^{4n})\right\}. \quad ...(2.14)$$

The values of $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ of (ii) when substituted in (2.1) yield

$$1 + \sum_{n=1}^{\infty} [\rho_1]_n [\rho_2]_n \frac{[aq^3]_{n-1,q^3}}{[q]_n [aq]_{2n-1}} \left(\frac{a}{\rho_1\rho_2 q^m}\right)^n = \frac{[aq/\rho_1]_\infty [aq/\rho_2]_\infty}{[aq]_\infty [aq/\rho_1\rho_2]_\infty}$$

$$\times \left\{ {}_3\phi_1 \left[\begin{matrix} \rho_1, \rho_2, q^{-m-1}; q \\ \rho_1\rho_2/a \end{matrix}\right]\right.$$

$$+ \sum_{n=1}^{\infty} (-)^n \frac{(1 - aq^{6n}) [aq^3]_{n-1,q^3} [\rho_1]_{3n} [\rho_2]_{3n}}{[q^3]_{n,q^3} [aq/\rho_1]_{3n} [aq/\rho_2]_{3n}}$$

$$\left. \times \frac{a^{4n} q^{3n(3n-2m-1)/2}}{\rho_1^{3n} \rho_2^{3n}} \, {}_3\phi_1 \left[\begin{matrix} \rho_1 q^{3n}, \rho_2 q^{3n}, q^{-m-1}; q \\ \rho_1\rho_2/a \end{matrix}\right]\right\}. \quad ...(2.15)$$

(2.15) on $\rho_1, \rho_2 \to \infty$ yields

$$1 + \sum_{n=1}^{\infty} \frac{[aq^3]_{n-1,q^3}}{[q]_n [aq]_{2n-1}} a^n q^{n(n-m-1)} = \frac{1}{[aq]_\infty} \left[\sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} a^s q^{s(s+1)/2}\right.$$

$$+ \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} a^s q^{s(s+1)/2} \sum_{n=1}^{\infty} (-)^n \frac{[aq^3]_{n-1,q^3} (1 - aq^{6n})}{[q^3]_{n,q^3}} a^{4n}$$

$$\left. \times q^{3n(9n-2m+4s-3)/2}\right]. \quad ...(2.16)$$

Setting $a = q^3$ in (2.16), we have

$$[q]_\infty \sum_{n=0}^{\infty} \frac{[q^3]_{n,q^3} q^{n(n-m+2)}}{[q]_n [q]_{2n+2}} = \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^{s(s+7)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{27n-3m+6s-6})(1 - q^{27n+3m-6s-21})(1 - q^{27n})\right\}.$$

...(2.17)

For $m = -1$, 0 and 1 (2.17) gives three known identities of Rogers-Ramanujan type related to the modulus 27 [Slater 1952, eqn. (90), (91) and (92)].

Setting $m = 2$ in (2.17) and using eqn. (91) of Slater (1952) we get on some simplification that

$$\sum_{n=0}^{\infty} \frac{[q^3]_{n,q^3}\, q^{n^2}}{[q]_n\, [q]_{2n+2}} \{1 - (1+q)\, q^{2n+1}\} = \prod_{n \not\equiv 0,\, 12,\, 15 \pmod{27}} (1 - q^n)^{-1}. \quad \ldots(2.18)$$

However, setting $a = 1$ in (2.16), we could have also obtained

$$1 + \sum_{n=1}^{\infty} \frac{[q^3]_{n-1,q^3}\, q^{n(n-m-1)}}{[q]_n\, [q]_{2n-1}} = \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s\, [q]_\infty} q^{s(s+1)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{27n-3m+6s-18})\,(1 - q^{27n+3m-6s-9})\,(1 - q^{27n})\right\} \quad \ldots(2.19)$$

which for $m = -1$ yields a known identity on modulus 27 [Slater 1952, eqn. (93)].

Next, in (2.16) letting $\rho_2 \to \infty$, $\rho_1 = -q^2$, $a = q^3$, we get an identity on modulus 18:

$$\sum_{n=0}^{\infty} \frac{[-q]_{n+1}\, [q^3]_{n,q^3}\, q^{n(n-2m+1)/2}}{[q]_n\, [q]_{2n+2}} = \frac{[-q]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s\, [q^{-s}]_j}{[q]_s\, [q]_j}$$

$$\times\, q^{sj+2s+2j} \prod_{n=1}^{\infty} \left\{(1 - q^{18n-3m+3j+3s-6})(1 - q^{18n+3m-3j-3s-12})(1 - q^{18n})\right\} \quad \ldots(2.20)$$

which for $m = -1$ yields a known identity [Slater 1952, eqn. (76)].

For $m = 0$, we have from (2.20) on using eqn. (76) of Slater (1952)

$$\sum_{n=0}^{\infty} \frac{[-q]_{n+1}\, [q^3]_{n,q^3}}{[q]_n\, [q]_{2n+2}}\, q^{n(n+1)/2}$$

$$= [-q]_\infty \prod_{n \not\equiv 0,\, 6,\, 12 \pmod{18}} (1 - q^n)^{-1} \quad \ldots(2.21)$$

and for $m = 1$, we get

$$\sum_{n=0}^{\infty} \frac{[-q]_{n+1}\,[q^3]_{n,q^3}}{[q]_n\,[q]_{2n+2}}\, q^{n(n-1)/2}\,\{1 - q^n(1+q)(1+q^{n+1})\}$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-q^{18n-9})^2\,(1-q^{18n})}{(1-q^n)\,(1-q^{2n-1})}\right] = \prod_{n=1}^{\infty}\left[\frac{(1-q^{9n})\,(1-q^{18n-9})}{(1-q^n)\,(1-q^{2n-1})}\right]. \qquad \text{...(2.22)}$$

However, in (2.16) letting $\rho_2 \to \infty$, $\rho_1 = -q^{3/2}$, $a = q^3$, we get on some simplification (on replacing q by q^2) the following identity on modulus 36:

$$\sum_{n=0}^{\infty} \frac{[q^6]_{n,q^6}\, q^{n(n-2m+2)}}{[q]_{2n+1}\,[q^4]_{n+1,q^4}} = \sum_{j=0}^{m+1}\sum_{s=j}^{m+1} (-)^{s+j}\, \frac{[q^{-2m-2}]_{s,q^2}\,[q^{-2s}]_{j,q^2}}{[q^2]_{s,q^2}\,[q^2]_{j,q^2}}\, q^{3j+5s+2sj}$$

$$\times \prod_{n=1}^{\infty}\left[\frac{(1-q^{36n-6m+6j+6s-9})\,(1-q^{36n+6m-6j-6s-27})\,(1-q^{36n})}{(1-q^{4n})\,(1-q^{2n-1})}\right]. \qquad \text{...(2.23)}$$

For $m = -1$, this yields a known identity [Slater 1952, eqn. (116)].

For $m = 0$, (2.23) yields [on using eqn. (116) of Slater (1952)] :

$$\sum_{n=0}^{\infty} \frac{[q^6]_{n,q^6}\, q^{n(n+2)}}{[q^2]_{2n+1}[q^4]_{n+1,q^4}} = \prod_{n=1}^{\infty}\left[\frac{(1-q^{36n-9})\,(1-q^{36n-27})\,(1-q^{36n})}{(1-q^{4n})\,(1-q^{2n-1})}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-q^{18n-9})\,(1-q^{36n})}{(1-q^{4n})\,(1-q^{2n-1})}\right]. \qquad \text{...(2.24)}$$

Whereas for $m = 1$, (2.23) gives

$$\sum_{n=0}^{\infty} \frac{[q^6]_{n,q^6}\, q^{n^2}(1 - q^{2n+1} - q^{4n+4})}{[q]_{2n+1}\,[q^4]_{n+1,q^4}}$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-q^{36n-21})\,(1-q^{36n-15})\,(1-q^{36n})}{(1-q^{4n})\,(1-q^{2n-1})}\right]. \qquad \text{...(2.25)}$$

Now, if we let $\rho_2 \to \infty$, $\rho_1 = -1$, $a = 1$ in (2.15), we get

$$1 + \sum_{n=1}^{\infty} \frac{[-1]_n[q^3]_{n-1,q^3}q^{n(n-2m-1)/2}}{[q]_n\,[q]_{2n-1}} = \frac{[-q]_\infty}{[q]_\infty}\left[\sum_{s=0}^{m+1} (-)^s\, \frac{[-1]_s[q^{-m-1}]_s}{[q]_s}\, q^s + \right.$$

(equation continued on p. 779)

$$+ 2 \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^s \sum_{n=1}^{\infty} (-)^n [-q^{3n}]_s \, q^{n(9n-3m+3s-3)} \Big]. \qquad \ldots(2.26)$$

Let

$$S = \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^s \sum_{n=1}^{\infty} (-)^n q^{3n(3n-m+s-1)} [-q^{3n}]_s$$

$$= \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^s \sum_{j=0}^{s} (-)^j \frac{[q^{-s}]_j}{[q]_j} q^{sj} \sum_{n=1}^{\infty} (-)^n q^{3n(3n-m+s-1+j)}$$

$$= \sum_{r=0}^{m+1} (-)^{m+1-r} \frac{[q^{-m-1}]_{m+1-r}}{[q]_{m+1-r}} q^{m+1-r}$$

$$\times \sum_{j=0}^{m+1-r} (-)^j \frac{[q^{-m-1+r}]_j}{[q]_j} q^{(m+1-r)j} \sum_{n=-\infty}^{-1} (-)^n q^{3n(3n+r-j)} \qquad \ldots(2.27)$$

(on setting $s = m + 1 - r$ and replacing in n by $-n$).

$$= \sum_{r=0}^{m+1} \frac{q^{-[(m+1-r)(m+r)+2r(m+2)-r(r+1)]/2}}{[q]_r}$$

$$\times \sum_{k=r}^{m+1} (-)^k \frac{[q^{-m-1}]_k}{[q]_{k-r}} q^{(k-r)(m+1-r)} \sum_{n=-\infty}^{-1} (-)^n q^{3n(3n+2r-k)}$$

$$= \sum_{r=0}^{m+1} \sum_{l=r}^{m+1} (-)^{r+l} \frac{[q^{-m-1}]_l \, [q^{-l}]_r}{[q]_l \, [q]_r} q^{r(1+l)} \sum_{n=-\infty}^{-1} (-)^n q^{3n(3n+r-m+l-1)} \qquad \ldots(2.28)$$

(on setting $k = m + 1 + r - l$). In view of (2.27) and (2.28), we have

$$2S = \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s \, [q^{-s}]_j}{[q]_s \, [q]_j} q^{s(1+j)} \sum_{n=-\infty}^{\infty}{}' (-)^n q^{3n(3n-m+j+s-1)}$$

where the (′) in Σ denotes the term corresponding to $n = 0$, is deleted.

Or

$$2S = \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s \, [q^{-s}]_j}{[q]_s \, [q]_j} \, q^{s(1+j)} \sum_{n=-\infty}^{\infty} (-)^n \, q^{3n(3n-m+j+s-1)}$$

$$- \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s \, [-1]_s}{[q]_s} \, q^s ; \left(\text{because } {}_1\phi_0 \left[\begin{matrix} q^{-s}; \, -q^s \\ - \end{matrix} \right] = [-1]_s \right), \qquad \ldots(2.29)$$

Using (2.29) in (2.26), we get

$$1 + 2 \sum_{n=1}^{\infty} \frac{[-q]_{n-1} \, [q^3]_{n-1,q^3}}{[q]_n \, [q]_{2n-1}} \, q^{n(n-2m-1)/2}$$

$$= \frac{[-q]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s \, [q^{-s}]_j}{[q]_s \, [q]_j} \, q^{s(1+j)}$$

$$\times \prod_{n=1}^{\infty} \left\{ (1 - q^{18n-3m+3j+3s-12})(1 - q^{18n+3m-3j-3s-6})(1 - q^{18n}) \right\}. \qquad \ldots(2.30)$$

For $m = -1$, (2.30) gives the result

$$1 + 2 \sum_{n=1}^{\infty} \frac{[-q]_{n-1} \, [q^3]_{n-1,q^3}}{[q]_n \, [q]_{2n-1}} \, q^{n(n+1)/2} = \prod_{n=1}^{\infty} \left[\frac{(1 - q^{18n-9})^2 (1 - q^{18n})}{(1 - q^{2n-1})(1 - q^n)} \right]$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1 - q^{9n})(1 - q^{18n-9})}{(1 - q^{2n-1})(1 - q^n)} \right]. \qquad \ldots(2.31)$$

In view of (2.22), (2.27) yields the transformation

$$\sum_{n=0}^{\infty} \frac{[-q]_{n+1} \, [q^3]_{n,q^3}}{[q]_n \, [q]_{2n+2}} \, q^{n(n-1)/2} = 1 + 2 \sum_{n=1}^{\infty} \frac{[-q]_{n-1} \, [q^3]_{n-1,q^3}}{[q]_n \, [q]_{2n-1}} \, q^{n(n+1)/2}. \qquad \ldots(2.32)$$

Lastly, if in (2.16), we let $\rho_2 \to \infty$, $\rho_1 = q^{7/2}$, $a = q^3$, we get

$$1 + \sum_{n=1}^{\infty} (-)^n \frac{[q^2]_n \, [q^6]_{n-1,q^3}}{[q]_n \, [q^4]_{2n-1}} \, q^{n(n-2m+1)/2} =$$

(*equation continued on p.* 781)

$$= (1 - q^2) \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} \frac{[q^{-m-1}]_s\, [q^{-s}]_j}{[q]_s\, [q]_j}\, q^{sj+2s+2j}$$

$$\times \left\{ \sum_{n=0}^{\infty} q^{3n(3n-m+s+j+1)} - \sum_{n=1}^{\infty} q^{3n(3n+m-s-j-1)} \right\}. \quad \ldots(2.33)$$

For $m = -1$, (2.33) gives

$$\sum_{n=0}^{\infty} (-)^n \frac{[q^3]_{n,q^3}\, q^{n(n+3)/2}}{(1+q^{n+1})\, [q]_{2n+1}} = \sum_{n=0}^{\infty} q^{9n^2+6n} - \sum_{n=1}^{\infty} q^{9n^2-6n}. \quad \ldots(2.34)$$

In (2.33) setting $m = 0$ and using (2.34), we get

$$\sum_{n=0}^{\infty} (-)^n \frac{[q^3]_{n,q^3}}{[q]_{2n+1}}\, q^{n(n+1)/2} = \sum_{n=0}^{\infty} q^{9n^2+3n} - \sum_{n=1}^{\infty} q^{9n^2-3n}. \quad \ldots(2.35)$$

The values of $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ of (iii) when substituted in (2.1) yield:

$$\sum_{n=0}^{\infty} \frac{[\rho_1]_{n,q^2}\, [\rho_2]_{n,q^2}\, [-(aq/b)]_{2n}}{[q^2]_{n,q^2}\, [-aq]_{2n}\, [a^2q^2/b^2]_{n,q^2}} \left(\frac{a^2}{\rho_1 \rho_2 q^{2m}} \right)^n$$

$$= \frac{[a^2q^2/\rho_1]_{\infty,q^2}\, [a^2q^2/\rho_2]_{\infty,q^2}}{[a^2q^2]_{\infty,q^2}\, [a^2q^2/\rho_1\rho_2]_{\infty,q^2}} \cdot \left\{ {}_3\phi_1^{(q^2)} \left[\begin{matrix} \rho_1, \rho_2, q^{-2m-2}; q^2 \\ \rho_1\rho_2/a^2 \end{matrix} \right] \right.$$

$$+ \sum_{s=0}^{m+1} \frac{[q^{-2m-2}]_{s,q^2}\, q^{2s}}{[q^2]_{s,q^2}\, [\rho_1\rho_2/a^2]_{s,q^2}}$$

$$\left. \times \sum_{n=1}^{\infty} \frac{[aq]_{n-1}\, (1 - aq^{2n})\, [b]_n\, [\rho_1]_{n+s,q^2}\, [\rho_2]_{n+s,q^2}}{[q]_n\, [(a/b)\, q]_n\, [(a^2/\rho_1)\, q^2]_{n,q^2}\, [(a^2/\rho_2)\, q^2]_{n,q^2}} \frac{a^{3n} q^{n^2-2mn}}{(b\rho_1\rho_2)^n} \right\}. \quad \ldots(2.36)$$

In (2.36) letting $b, \rho_1, \rho_2 \to \infty$ and $a = 1$, we get on some simplification

$$\sum_{n=0}^{\infty} \frac{q^{2n(n-m-1)}}{[q^2]_{n,^2q}\, [-q]_{2n}} = \frac{1}{[q]_\infty\, [-q]_\infty} \sum_{s=0}^{m+1} (-)^s \frac{[q^{-2m-2}]_{s,q^2}}{[q^2]_{s,q^2}}\, q^{s(s+1)}$$

$$\times \prod_{n=1}^{\infty} \left\{ (1 - q^{7n-2m+4s-6})\, (1 - q^{7n+2m-4s-1})\, (1 - q^{7n}) \right\}. \quad \ldots(2.37)$$

For $m = -1$, (2.37) yields a known identity [Slater 1952, eqn. (33)], whereas setting $m = 1$ in (2.37), we get

$$\frac{1}{q} \prod_{n \not\equiv 0, 1, 6 \pmod 7} (1 - q^n)^{-1} = \frac{1 + q^2}{q^2} \prod_{n \not\equiv 0, 3, 4 \pmod 7} (1 - q^n)^{-1}$$

$$- [-q]_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n-2)}}{[q^2]_{n,q^2} [-q]_{2n}}$$

$$= \frac{[-q]_\infty}{q^2} \left[(1 + q^2) \sum_{n=0}^{\infty} \frac{q^{2n^2}}{[q^2]_{n,q^2} [-q]_{2n}} - q^2 \sum_{n=0}^{\infty} \frac{q^{2n(n-2)}}{[q^2]_{n,q^2} [-q]_{2n}} \right]$$

$$= \frac{[-q]_\infty}{q^2} \left[\sum_{n=0}^{\infty} \frac{q^{2n^2}}{[q^2]_{n,q^2} [-q]_{2n}} - q^2 \sum_{n=1}^{\infty} \frac{q^{2n(n-2)} (1 - q^{4n})}{[q^2]_{n-1,q^2} [-q]_{2n-1}} \right].$$

$$\therefore \prod_{n \not\equiv 0, 1, 6 \pmod 7} (1 - q^n)^{-1} = [-q]_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n+1}} \qquad \ldots(2.38)$$

which is a known identity [Slater 1952, eqn. (31)].

Next, setting $m = 0$ in (2.37) and using (2.38), we get

$$\prod_{n \not\equiv 0, 2, 5 \pmod 7} (1 - q^n)^{-1} = [-q]_\infty \left\{ \sum_{n=0}^{\infty} \frac{q^{2n(n-1)}}{[q^2]_{n,q^2} [-q]_{2n}} \right.$$

$$\left. - \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n+1}} \right\}$$

$$= [-q]_\infty \left\{ \sum_{n=0}^{\infty} \frac{q^{2n(n-1)}}{[q^2]_{n,q^2} [-q]_{2n}} - \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n}} \right.$$

$$\left. + \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n}} - \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n+1}} \right\}$$

$$= [-q]_\infty \left\{ \sum_{n=1}^{\infty} \frac{q^{2n(n-1)}}{[q^2]_{n-1,q^2} [-q]_{2n-1}} + \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n}} - \right.$$

(equation continued on p. 783)

$$- \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n+1}}\Bigg\}$$

$$= [-q]_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q^2]_{n,q^2} [-q]_{2n}}. \qquad \ldots(2.39)$$

(2.39) is a known identity [Slater 1952, eqn. (32)].

Next, if in (2.36) we let $\rho_1, \rho_2 \to \infty$, $b \to 0$, $a = 1$, we get

$$[-q]_\infty \sum_{n=0}^{\infty} \frac{q^{n(3n-2m-2)}}{[q^2]_{n,q^2} [-q]_{2n}} = \frac{1}{[q]_\infty} \sum_{s=0}^{m+1} (-)^s \frac{[q^{-2m-2}]_{s,q^2}}{[q^2]_{s,q^2}} q^{s(s+1)}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{5n-2m+4s-5}) (1 - q^{5n+2m-4s}) (1 - q^{5n})\right\}. \qquad \ldots(2.40)$$

For $m = -1$, (2.40) yields eqn. (19) of Slater (1952) whereas for $m = 0$, (2.40) gives eqn. (15) of Slater (1952).

Using the values of $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ given by (iv) in (2.1), we get

$$\sum_{n=0}^{\infty} \frac{[aq]_{3n} [\rho_1]_{n,q^3} [\rho_2]_{n,q^3}}{[q^3]_{n,q^3} [a^3q^3]_{2n,q^3}} \left(\frac{a^3}{\rho_1\rho_2 q^{3m}}\right)^n = \frac{[(a^3/\rho_1) q^3]_{\infty,q^3} [(a^3/\rho_2) q^3]_{\infty,q^3}}{[a^3q^3]_{\infty,q^3} [(a^3/\rho_1\rho_2) q^3]_{\infty,q^3}}$$

$$\times \left\{ {}_3\phi_1^{(q^3)} \left[\begin{matrix} \rho_1, \rho_2 q^{-3m-3}; q^3 \\ \rho_1\rho_2/a^3 \end{matrix} \right] \right.$$

$$+ \sum_{n=1}^{\infty} (-)^n (1 - aq^{2n}) \frac{[aq]_{n-1} [\rho_1]_{n,q^3} [\rho_2]_{n,q^3}}{[q]_n [(a^3/\rho_1) q^3]_{n,q^3} [(a^3/\rho_2) q^3]_{n,q^3}}$$

$$\times \left. \left(\frac{a^{4n}}{\rho_1\rho_2 q^{3m}}\right)^n q^{n(3n-1)/2} \, {}_3\phi_1^{(q^3)} \left[\begin{matrix} \rho_1 q^{3n}, \rho_2 q^{3n}, q^{-3m-3}; q^3 \\ \rho_1\rho_2/a^3 \end{matrix} \right] \right\}. \qquad \ldots(2.41)$$

In (2.41) letting $\rho_1, \rho_2 \to \infty$, $a = 1$, we get

$$\sum_{n=0}^{\infty} \frac{[q]_{3n}\, q^{3n(n-m-1)}}{[q^3]_{n,q^3} [q^3]_{2n,q^3}} = \frac{1}{[q^3]_{\infty,q^3}} \sum_{s=0}^{m+1} (-)^s \frac{[q^{-3m-3}]_{s,q^3}}{[q^3]_{s,q^3}} q^{3s(s+1)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{9n-3m+6s-8}) (1 - q^{9n+3m-6s-1}) (1 - q^{9n})\right\}. \qquad \ldots(2.42)$$

For $m = -1$, (2.42) yields a known identity of Rogers-Ramanujan type [Slater 1952, eqn. (42)]. However in (2.41) letting $\rho_1, \rho_2 \to \infty$, $a = q$, we get

$$\sum_{n=0}^{\infty} \frac{[q]_{3n+1}\, q^{3n(n-m)}}{[q^3]_{n,q^3}\,[q^3]_{2n+1,q^3}} = \frac{1}{[q^3]_{\infty,q^3}} \sum_{s=0}^{m+1} (-)^s \frac{[q^{-3m-3}]_{s,q^3}}{[q^3]_{s,q^3}}\, q^{3s(s+3)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{9n-3m+6s-4})\,(1 - q^{9n+3m-6s-5})\,(1 - q^{9n})\right\}. \qquad \ldots(2.43)$$

(2.43) yields for $m = -1$ and $m = 0$ two of the known identities [Slater 1952, eqns. (40) and (41) respectively]. However from (2.42) and (2.43) we get the following equivalent product relations

$$\sum_{s=0}^{m} (-)^s \frac{[q^{-3m}]_{s,q^3}}{[q^3]_{s,q^3}}\, q^{3s(s+1)/2} \prod_{n=1}^{\infty} (1 - q^{9n-3m+6s-5})\,(1 - q^{9n+3m-6s-4})$$

$$= \sum_{s=0}^{m+1} (-)^s \frac{[q^{-3m-3}]_{s,q^3}}{[q^3]_{s,q^3}}\, q^{3s(s+1)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{9n-3m+6s-4})(1 - q^{9n+3m-6s-5})\right\}. \qquad \ldots(2.44)$$

On the other hand in (2.41) letting $\rho_2 \to \infty$, $\rho_1 = -q^{3/2}$, $a = 1$, we get on replacing q by q^2

$$\sum_{n=0}^{\infty} \frac{[q^2]_{3n,q^2}\,[-q^3]_{n,q^6}}{[q^6]_{n,q^6}\,[q^6]_{2n,q^6}}\, q^{3n(n-2m-2)}$$

$$= \frac{[-q^3]_{\infty,q^6}}{[q^6]_{\infty,q^6}} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-6m-6}]_{s,q^6}}{[q^6]_{s,q^6}} \frac{[q^{-6s}]_{j,q^6}}{[q^6]_{j,q^6}}\, q^{3(s+j+2sj)}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{12n-6m+6s+6j-13})\,(1 - q^{12n+6m-6s-6j+1})\,(1 - q^{12n})\right\}.$$

$$\ldots(2.45)$$

For $m = -1$, (2.45) yields

$$\sum_{n=0}^{\infty} \frac{[q^2]_{3n,q^2}\, q^{3n^2}}{[q^{12}]_{n,q^{12}}\,[q^3]_{2n,q^3}} = \frac{[q]_{\infty}\,[-q^3]_{\infty,q^6}}{[q^6]_{\infty,q^6}} \prod_{n \not\equiv 0,\,5,\,7 \pmod{12}} (1 - q^n)^{-1}.$$

$$\ldots(2.46)$$

Setting $m = 0$ in (2.45) and using (2.46), we get on some reduction

$$\sum_{n=0}^{\infty} \frac{[q^2]_{3n+1,q^2}\, q^{3n_2}}{[q^{12}]_{n,q^{12}}\, [q^3]_{2n,q^3}\, (q^{6n} - q^2)}$$

$$= \frac{[q]_\infty\, [-q^3]_{\infty,q^6}}{[q^6]_{\infty,q^6}} \prod_{n \not\equiv 0,\, 1,\, 11 \pmod{12}} (1 - q^n)^{-1}. \qquad \ldots(2.47)$$

Lastly, in (2.41) letting $\rho_2 \to \infty$, $\rho_1 = -q^2$, $a = q$, we get

$$\sum_{n=0}^{\infty} \frac{[q]_{3n+1}\, [-q^3]_{n,q^3} \,.\, q^{3n(n-2m)/2}}{[q^3]_{n,q^3}\, [q^3]_{2n+1,q^3}}$$

$$= \frac{[-q^3]_{\infty,q^3}}{[q^6]_{\infty,q^3}} \sum_{j=0}^{m+1} \sum_{s=0}^{m+1} (-)^{s+j} \frac{[q^{-3m-3}]_{s,q^3}\, [q^{-3m}]_{j,q^3}}{[q^3]_{s,q^3}\, [q^3]_{j,q^3}}\, q^{3(s+j+sj)}$$

$$\times \prod_{n=1}^{\infty} \left\{ (1 - q^{6n-3m+3s+3j-4})\, (1 - q^{6n+3m-3s-3j-2})\, (1 - q^{6n}) \right\}. \qquad \ldots(2.48)$$

For $m = -1$, (2.48) yields

$$\sum_{n=0}^{\infty} \frac{[q]_{3n+1}\, q^{3n(n+1)/2}}{\{[q^3]_{n,q^3}\}^2\, [q^3]_{n+1,q^6}} = \frac{[q]_\infty}{[q^6]_{\infty,q^3}\, [q^3]_{\infty,q^6}} \prod_{n \not\equiv 0,\, 1,\, 5 \pmod 6} (1 - q^n)^{-1}. \qquad \ldots(2.49)$$

Setting $m = 0$ in (2.48) and using (2.47), we get

$$\sum_{n=0}^{\infty} \frac{[q^2]_{3n+3}\, q^{3n(n+1)/2}}{[q^3]_{n,q^3}\, [q^3]_{n+1,q^3}\, [q^3]_{n+2,q^6}}$$

$$= \frac{[q]_\infty}{[q^6]_{\infty,q^3}\, [q^3]_{\infty,q^6}} \prod_{n \not\equiv 0,\, 2,\, 4 \pmod 6} (1 - q^n)^{-1}.$$

The values of $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ of (v) when substituted in (2.1) yield:

$$\sum_{n=0}^{\infty} \frac{[aq/f]_{n,q^2}\, [\rho_1]_n\, [\rho_2]_n}{[aq]_{n,q^2}\, [q]_n\, [aq/f]_n} \left(\frac{a}{\rho_1 \rho_2 q^m} \right)^n$$

$$= \frac{[aq/\rho_1]_\infty\, [aq/\rho_2]_\infty}{[aq]_\infty\, [aq/\rho_1\rho_2]_\infty} \left\{ {}_3\phi_1 \left[\begin{matrix} \rho_1,\, \rho_2,\, q^{-m-1};\, q \\ \rho_1\rho_2/a \end{matrix} \right] + \right.$$

(*equation continued on p.* 786)

$$+\sum_{n=1}^{\infty}\frac{(1-aq^{4n})\,[aq^2]_{n-1,q^2}\,[f]_{n,q^2}\,[\rho_1]_{2n}\,[\rho_2]_{2n}}{[q^2]_{n,q^2}\,[aq^2/f]_{n,q^2}\,[aq/\rho_1]_{2n}\,[aq/\rho_2]_{2n}}$$

$$\times\left(\frac{a^3q^{2n-2m}}{f\rho_1^2\rho_2^2}\right)^n {}_3\phi_1\left[\begin{matrix}\rho_1q^{2n},\ \rho_2q^{2n},\ q^{-m-1};\ q\\ \rho_1\rho_2/a\end{matrix}\right]\Bigg\}. \qquad \ldots(2.50)$$

In (2.50) letting $\rho_2\to\infty$, $f\to 0$, $\rho_1=-q^{1/2}$, $a=1$, we get

$$\sum_{n=0}^{\infty}\frac{q^{n(2n-2m-3)/2}}{[q^{1/2}]_{2n,q^{1/2}}}=\frac{[-q^{1/2}]_\infty}{[q]_\infty}\sum_{j=0}^{m+1}\sum_{s=j}^{m+1}\frac{[q^{-m-1}]_s\,[q^{-s}]_j}{[q]_s\,[q]_j}\,q^{(2sj+s+j)/2}$$

$$\times\prod_{n=1}^{\infty}\Big\{(1-q^{6n-2m+2s+2j-6})\,(1-q^{6n+2m-2s-2j})\,(1-q^{6n})\Big\}. \qquad \ldots(2.51)$$

In (2.50) letting $\rho_2\to\infty$, $f\to 0$, $\rho_1=q^{1/2}$, $a=1$, we get

$$\sum_{n=0}^{\infty}\frac{(-)^n\,q^{n(2n-2m-3)/2}}{[q]_n\,[-q^{1/2}]_n}=\frac{[q^{1/2}]_\infty}{[q]_\infty}\sum_{j=0}^{m+1}\sum_{s=j}^{m+1}\frac{[q^{-m-1}]_s\,[q^{-s}]_j}{[q]_s\,[q]_j}\,q^{(2sj+s+j)/2}$$

$$\times\prod_{n=1}^{\infty}\Big\{(1-q^{6n-6-2m+2s+2j})\,(1-q^{6n+2m-2s-2j})\,(1-q^{6n})\Big\}. \qquad \ldots(2.52)$$

On the other hand in (2.50) letting $\rho_2\to 0$, $f\to 0$, $\rho_1=-q^{3/2}$, $a=q$, we get

$$\sum_{n=0}^{\infty}\frac{q^{n(2n-2m-1)/2}}{[q]_{2n,q^{1/2}}}=\frac{[-q^{3/2}]_\infty}{[q^2]_\infty}\sum_{j=0}^{m+1}\sum_{s=j}^{m+1}(-)^{s+j}\frac{[q^{-m-1}]_s\,[q^{-s}]_j}{[q]_s\,[q]_j}\,q^{(3s+3j+2sj)/2}$$

$$\times\prod_{n=1}^{\infty}\Big\{(1-q^{6n-4-2m+2s+2j})\,(1-q^{6n-2+2m-2s-2j})\,(1-q^{6n})\Big\}. \qquad \ldots(2.53)$$

Next, in (2.50) letting $\rho_2\to 0$, $f\to 0$, $\rho_1=q^{3/2}$, $a=q^2$, we get

$$\sum_{n=0}^{\infty}\frac{(-)^n\,q^{n(2n-2m-1)/2}}{[q]_n\,[-q^{3/2}]_n}=\frac{[q^{3/2}]_\infty}{[q^2]_\infty}\sum_{j=0}^{m+1}\sum_{s=j}^{m+1}\frac{[q^{-m-1}]_s\,[q^{-s}]_j}{[q]_s\,[q]_j}\,q^{(3s+3j+2sj)/2}$$

$$\times\prod_{n=1}^{\infty}\Big\{(1-q^{6n-4-2m+2s+2j})\,(1-q^{6n-2+2m-2s-2j})\,(1-q^{6n})\Big\}. \qquad \ldots(2.54)$$

Further, if in (2.50) letting $\rho_2, f \to 0$, $\rho_1 = -1$, $a = 1$, we get

$$\sum_{n=0}^{\infty} \frac{[-1]_n\, q^{n(n-m-1)}}{[q]_n\, [q]_{n,q^2}} = \frac{[-q]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s\, [q^{-s}]_j}{[q]_s\, [q]_j}\, q^{s(1+j)}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{6n-2m+2s+2j-5})(1 - q^{6n+2m-2s-2j-1})\,(1 - q^{6n})\right\}. \quad \ldots(2.55)$$

For $m = -1$, (2.55) yields

$$\sum_{n=0}^{\infty} \frac{[-1]_n}{[q]_n\, [q]_{n,q^2}}\, q^{n^2} = \frac{[-q]_\infty}{[q]_\infty} \prod_{n=1}^{\infty} \left\{(1 - q^{6n-3})^2\, (1 - q^{6n})\right\}. \quad \ldots(2.56)$$

Setting $m = 0$, in (2.55) and using (2.56), we get

$$\sum_{n=0}^{\infty} \frac{[-q]_n\, q^{n(n+1)}}{[q]_n\, [q]_{n+1,q^2}} = [-q]_\infty \prod_{n\not\equiv 0,\,1,\,5\,(\mathrm{mod}\ 6)} (1 - q^n)^{-1}. \quad \ldots(2.57)$$

Setting $m = -1$ in (2.51) – (2.54), we get

$$\prod_{n\not\equiv 0,\,2,\,4\,(\mathrm{mod}\ 6)} (1 - q^n)^{-1} = \frac{1}{[-q^{1/2}]_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n-1)/2}}{[q^{1/2}]_{2n,q^{1/2}}}$$

$$= \frac{1}{[q^{1/2}]_\infty} \sum_{n=0}^{\infty} \frac{(-)^n\, q^{n(2n-1)/2}}{[q]_n\, [-q^{1/2}]_n}$$

$$= \frac{[-q]_\infty}{[-q^{3/2}]_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)/2}}{[q]_n\, [-q^{3/2}]_n}$$

$$= \frac{[q^6]_{\infty,q^3}\, [q^3]_{\infty,q^6}}{[q]_\infty} \sum_{n=0}^{\infty} \frac{[q^3]_{3n+3}\, q^{3n(n+1)/2}}{[q^3]_{n,q^3}\, [q^3]_{n+1,q^3}\, [q^3]_{n+2,q^6}} \quad \ldots(2.58)$$

in view of (2.49).

Next, in (2.50) letting $\rho_1, \rho_2, f \to \infty$, $a = 1$, gives

$$\sum_{n=0}^{\infty} \frac{q^{n(n-m-1)}}{[q]_n\, [q]_{n,q^2}} = \frac{1}{[q]_\infty} \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s}\, q^{s(s+1)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 - q^{14n-10-2m+4s})\,(1 - q^{14n-4+2m-4s})\,(1 - q^{14n})\right\}. \quad \ldots(2.59)$$

(2.59), for $m = -1, 0$, reduces to the two known identities [Slater 1952, eqns. (61) and (60)].

Setting $m = 1$ in (2.59) and using eqns. (61) and (60) of Slater (1952), we get [Slater 1952, eqn. (59)]:

$$\prod_{n \not\equiv 0, 2, 12 \pmod{14}} (1 - q^n)^{-1} = \sum_{n=0}^{\infty} \frac{q^{n(n-2)}}{[q]_n [q]_{n,q^2}}$$

$$- \frac{1+q}{q} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[q]_n [q]_{n,q^2}} - \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{[q]_n [q]_{n+1,q^2}}$$

$$= \sum_{n=1}^{\infty} \frac{(1-q^n)\, q^{n(n-2)}}{[q]_n [q]_{n,q^2}} - \frac{1}{q} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[q]_n [q]_{n,q^2}}$$

$$+ \sum_{n=1}^{\infty} \frac{(1-q^n)\, q^{n(n-1)}}{[q]_n [q]_{n,q^2}} - \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[q]_n [q]_{n+1,q^2}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{[q]_n [q]_{n+1,q^2}}. \qquad \ldots(2.60)$$

In (2.50) letting $\rho_1, \rho_2 \to \infty$, $f \to 0$, $a = 1$, we get

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-m-3)/2}}{[q]_n [q]_{n,q^2}} = \frac{1}{[q]_\infty} \sum_{s=0}^{m+1} (-)^s \frac{[q^{-m-1}]_s}{[q]_s} q^{s(s+1)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{ (1 - q^{10n-8-2m+4s})(1 - q^{10n-2+2m-4s})(1 - q^{10n}) \right\}. \qquad \ldots(2.61)$$

For $m = -1, 0$, (2.61) yields two known identities [Slater 1952, eqn. (46) and (44)]. In (2.50) letting $\rho_2, f \to \infty$, $\rho_1 = -1$, $a = 1$, we get

$$\sum_{n=0}^{\infty} \frac{[-1]_n\, q^{(n-2m-1)/2}}{[q]_n [q]_{n,q^2}} = \frac{[-q]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{s(1+j)}$$

$$\times \prod_{n=1}^{\infty} \left\{ (1 - q^{10n-7-2m+2j+2s})(1 - q^{10n-3+2m-2j-2s})(1 - q^{10n}) \right\}.$$

$$\ldots(2.62)$$

For $m = -1$, (2.62) yields

$$\sum_{n=0}^{\infty} \frac{[-1]_n q^{n(n+1)/2}}{[q]_n [q]_{n,q^2}} = \prod_{n=1}^{\infty} \left\{ \frac{(1-q^{10n-5})^2 (1-q^{10n})}{(1-q^n)(1-q^{2n-1})} \right\}. \qquad ...(2.63)$$

In (2.62) setting $m = 0, 1$ and using (2.63), we get two known identities [Slater 1952, eqns. (45) and (43)]. On the other hand in (2.50) letting $\rho_2, f \to \infty$, $\rho_1 = -q$, $a = q^2$, we get [Slater 1952, eqn. (43)] directly by setting $m = -1$.

However, in (2.50) letting $\rho_2, f \to \infty$, $\rho_1 = q^{1/2}$, $a = 1$, we get

$$\sum_{n=0}^{\infty} \frac{(-)^n [q^{1/2}]_n}{[q]_n [q]_{n,q^2}} q^{n(n-2m-2)/2}$$
$$= \frac{[q^{1/2}]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{(s+j+2sj)/2}$$
$$\times \prod_{n=1}^{\infty} \left\{ (1-q^{10n-8-2m+2s+2j})(1-q^{10n-2+2m-2s-2j})(1-q^{10n}) \right\}. \qquad ...(2.64)$$

(2.64) for $m = -1$, yields

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n^2/2}}{[q]_n [-q^{1/2}]_n} = [q^{1/2}]_\infty \prod_{n \not\equiv 0,4,6 \pmod{10}} (1-q^n)^{-1}. \qquad ...(2.65)$$

In (2.64) letting $m = 0$ and using (2.65), we get

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+2)/2}}{(1-q^{n+(1/2)}) [q]_n [-q^{1/2}]_{n+1}} = [q^{1/2}]_\infty \prod_{n \not\equiv 0,2,8 \pmod{10}} (1-q^n)^{-1}. \qquad ...(2.66)$$

Furthermore, in (2.50) letting $\rho_2, f \to \infty$, $\rho_1 = -q^{1/2}$, $a = 1$, we have:

$$\sum_{n=0}^{\infty} \frac{q^{n(n-2m-2)/2}}{[q]_n [q^{1/2}]_n} = \frac{[-q^{1/2}]_\infty}{[q]_\infty} \sum_{j=0}^{m+1} \sum_{s=j}^{m+1} (-)^{s+j} \frac{[q^{-m-1}]_s [q^{-s}]_j}{[q]_s [q]_j} q^{(s+j+2sj)/2}$$
$$\times \prod_{n=1}^{\infty} \left\{ (1-q^{10n-8-2m+2j+2s})(1-q^{10n-2+2m-2s-2j})(1-q^{10n}) \right\}. \qquad ...(2.67)$$

(2.67) for $m = -1$, yields

$$\sum_{n=0}^{\infty} \frac{q^{n^2/2}}{[q]_n [q^{1/2}]_n} = [-q^{1/2}]_\infty \prod_{n \not\equiv 0, 4, 6 \,(\mathrm{mod}\ 10)} (1 - q^n)^{-1}. \qquad \text{...(2.68)}$$

Setting $m = 0$ in (2.67) and using (2.68) we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)/2}}{[q]_n [q^{1/2}]_{n+1}} = [-q^{1/2}]_\infty \prod_{n \not\equiv 0, 2, 8 \,(\mathrm{mod}\ 10)} (1 - q^n)^{-1}. \qquad \text{...(2.69)}$$

Lastly if we use (vi) in (2.1) and let $\rho_1, \rho_2 \to \infty$, $a = 1$, we get on some reduction:

$$[q]_\infty \sum_{n=0}^{\infty} \frac{q^{n(n-m-1)}}{[q]_{2n}} = -q^{1-m} \sum_{j=0}^{m+1} (-)^j \frac{[q^{-m-1}]_j}{[q]_j} q^{j(j+1)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 + q^{30n-7-3m+6j})(1 + q^{30n-23+3m-6j})(1 - q^{30n})\right\}$$

$$+ \sum_{j=0}^{m+1} (-)^j \frac{[q^{-m-1}]_j}{[q]_j} q^{j(j+1)/2}$$

$$\times \prod_{n=1}^{\infty} \left\{(1 + q^{30n-17-3m+6j})(1 + q^{30n-13+3m-6j})(1 + q^{30n})\right\}. \qquad \text{...(2.70)}$$

For $m = -1, 1, 0, 2$, (2.70) yields on some manipulation four identities related to modulus 30 proved by Slater [1952, eqns. (98), (96), (94) and (99) respectively]. On the other hand using the values of $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ of A(5) of Slater (1952) an identity can be written out from which the other three identities of Slater [1952, eqns. (95), (97), (100)] related to modulus 30 can be obtained on giving suitable values to m.

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