Indian J. pure appl. Math., 11(6) : 791–799, June 1980

NONLINEAR RANDOM EQUATIONS WITH P-COMPACT OPERATORS IN BANACH SPACES

MOHAN JOSHI

Mathematics Group, Birla Institute of Technology and Science, Pilani 333031

(*Received* 17 *September* 1979)

In this paper we study the existence of random solutions of operator equations involving P-compact operators. In our main theorem we get the random analogue of the deterministic results obtained by Petryshyn (1966) for P-compact operators. As corollaries we obtain the random analogues of fixed point theorems of Schauder, Rothe and Altman and also an existence theorem for random equations involving monotone operators.

1. INTRODUCTION

The study of random operator equations was first initiated by Spacek (1955) and Hans (1957, 1960) with a systematic investigation of random fixed point theorems. Now, there is relatively wide literature in this field with contributions by Bharucha-Reid (1972, 1976), Himmelberg (1975), Itoh (1978), Kannan and Salehi (1976), Kuratowskii and Ryll-Nardzewski (1965), Mukherjee (1966), Prakasa Rao (1973), Tsokos and Padgett (1974) and others.

In this paper we study the existence of random solutions of random operator equations involving P-compact operator. Our main purpose is to obtain the random analogues of the deterministic results obtained by Petryshyn (1966) for P-compact operators. In studying such a class of random nonlinear operators, we cover some important class of operators, namely, closed precompact random operators and demicontinuous monotone random operators. So our results synthesize almost all known results for such class of nonlinear random operators. In particular, we obtain as corollaries the random analogues of fixed point theorems of Schauder, Rothe and Altman. Also as a corollary we get the existence theorem for random equations involving monotone operators.

2. PRELIMINARIES

Throughout this paper (Ω, β, μ) denotes a probability measure space and X a separable Banch space with dual X^*. $\langle\ ,\ \rangle$ denotes the duality pairing between X and X^*. If X is a Hilbert space, then $\langle\ ,\ \rangle$ indicates the inner product in X. For any subset D of X, 2^D denotes the family of all subsets of D and $WK(D)$ the family of nonempty weakly compact subsets of D.

A mapping $T : \Omega \to 2^X$ is said to be (W_-) measurable if for any (weakly) closed subset C of X, $T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\}$ is in β. A (W_-) measurable mapping $g : \Omega \to X$ is said to be a measurable selector of a (W_-) measurable mapping T if for any $\omega \in \Omega$, $g(\omega) \in T(\omega)$. Any X-valued measurable function is called a random variable. We denote by $B(\Omega, X, \mu)$ the class of all X-valued measurable functions which are Bochner integrable. $B(\Omega, X, \mu)$ is a Banach space with norm defined by

$$\| x \|_{B(\Omega,X,\mu)} = \int_{\Omega} \| x(\omega) \| \, d\mu.$$

A necessary and sufficient condition that $x(\omega)$ be Bochner integrable is that $x(\omega)$ is measurable and $\int_{\Omega} \| x(\omega) \| \, d\mu < \infty$. An operator $T : X \to X$ is said to be bounded if it maps bounded sets into bounded sets. T is demi-continuous if $x_n \to x$ implies that $Tx_n \rightharpoonup Tx$ (weak convergence); T is closed if $x_n \to x$ and $Tx_n \to y$ imply $Tx = y$; T is weakly continuous if $x_n \rightharpoonup x$ implies that $Tx_n \rightharpoonup Tx$. T is compact if it maps bounded sets into precompact sets. If X is a Hilbert space, T is called monotone if $\langle Tx - Ty, x - y \rangle \geqslant 0$ for all x, y in X.

Let D be a subset of X, $T : \Omega \times D \to X$ is called a random operator if for any $x \in D$, T(.) x is measurable. An equation of the type T(.) $x(.) = y(.)$, where y is a given random variable, is called a random operator equation.

A X-valued random variable $x(\omega)$ which satisfies $\mu\{\omega : T(\omega)\, x(\omega) = y(\omega)\} = 1$, is said to be random solution of the above equation.

With these definitions we can obtain the following theorems.

Theorem 2.1 (Bharucha-Reid 1972) — Let x be a random variable with values in a separable Banach space X and let T be demi-continuous random operator of the space $\Omega \times X$ into a metric space Z. Then the mapping W of Ω into Z defined by, for every $\omega \in \Omega$, $W(\omega) = T(\omega)\, x(\omega)$ is a random variable with values in Z.

Theorem 2.2 (Bharucha-Reid 1976) — Let E be a compact convex subset of a separable Banach space X and $T(\omega)$ a continuous random operator mapping E into itself. Then there exists a random solution of the fixed point equation

$$T(\omega)\, x(\omega) = x(\omega).$$

Theorem 2.3 (Himmelberg 1975) — Let X be a separable metrizable space and let $G_n : \Omega \to 2^X$ be a sequence of weakly measurable functions with closed values. Also assume that for each $\omega \in \Omega$, $G_n(\omega)$ is compact for some n, then $G = \cap G_n$ is measurable.

In the following if U is a family of functions from Ω to X, then $U(\omega)$ denotes the set $\{u(\omega) : u \in U\}$ for each $\omega \in \Omega$.

Theorem 2.4 (Himmelberg 1975) — Let X be separable metric space and let $F : \Omega \to 2^X$ be a function with complete values. Then F is weakly measurable if there exists a countable family U of measurable selectors for F such that $F(\omega) = \overline{U(\omega)}$ for all $\omega \in \Omega$.

Theorem 2.5 (Kuratowski and Ryll-Nardzewski 1965) — Let X be a separable metric space and let $T : \Omega \to 2^X$ be a closed valued measurable function. Then there exists at least one random selector of T.

3. Random Equations in a Finite Dimensional Space

In this section we treat a random equation in a finite dimensional space and obtain the random analogue of Theorem 1 of the Petryshyn (1966). As a corollary we obtain the Proposition 3.1 of Itoh (1978) for random operators.

In what follows X is a finite dimensional space, B_r the closed ball in X and S_r its boundary.

Theorem 3.1 — Let $T : \Omega \times B_r \to X$ be a continuous random operator and $\mu(\omega)$ be any random constant. Then there exists a random solution $x(\omega)$ in B_r of the equation

$$T(\omega)\, x(\omega) = \mu(\omega)\, x(\omega) \qquad \text{...(3.1)}$$

provided that the operator $T(\omega)$ satisfies either the condition

($\leqslant$) If for some x in S_r the equation $T(\omega)\, x = \alpha x$ holds, then $\alpha \leqslant \mu(\omega)$ or the condition

($\geqslant$) If for the some x in S_r the equation $T(\omega)\, x = \alpha x$ holds, then $\alpha \geqslant \mu(\omega)$.

Proof : Assume that the operator $T(\omega)$ satisfies the condition ($\leqslant$). We then consider the random operator $C(\omega)\, x = T(\omega)\, x - \mu(\omega) + x$ and the retraction mapping $R : X \to B_r$ defined as

$$Rx = \begin{cases} x & \text{if } \| x \| \leqslant r \\ \dfrac{r\, x}{\| x \|} & \text{if } \| x \| \geqslant r. \end{cases} \qquad \text{...(3.2)}$$

The operator $C_1(\omega)x = RC(\omega)\, x$, being a composition of continuous operator and a continuous random operator, will be a continuous random operator; mapping B_r into itself. Since X is finite, B_r is compact and hence by Theorem 2.2 it follows that $C_1(\omega)$ has a random solution $x(\omega)$ in B_r. Proceeding as in Petryshyn (1966) one can now show that $x(\omega)$ is also a fixed point of $C(\omega)$ and this proves the result.

Similarly, one proves the theorem with condition ($\leqslant$) replaced by the condition ($\geqslant$). We need define only the new random operator $S(\omega)$ by $S(\omega)x = 2\mu(\omega)\, x - T(\omega)x$ and observe that $S(\omega)$ satisfies condition ($\leqslant$) iff $T(\omega)$ satisfies the condition ($\geqslant$).

We have the following analogue of Corollary 1 of Petryshyn (1966).

Corollary 3.1 — Let $T : \Omega \times B_r \to X$ be a random continuous operator such that

either $\quad \| x - T(\omega)\, x \|^2 \geqslant \| T(\omega)\, x \|^2 + \| x \|^2, \quad x \in S_r$ ...(3.3)

or $\quad \| x - T(\omega)\, x \|^2 \leqslant \| T(\omega)\, x \|^2 + \| x \|^2, \quad x \in S_r.$ ...(3.4)

Then there exists a random solution $x(\omega) \in B_r$ of the operator equation

$$T(\omega)\, x(\omega) = 0.$$

If X is a Hilbert space, then (3.4) is equivalent to the condition

$$\langle T(\omega)x, x \rangle \geqslant 0 \text{ for } \| x \| = r.$$

As a corollary, we therefore obtain the Proposition 3.1 of Itoh (1978) without the condition of monotonicity on $T(\omega)$.

Corollary 3.2 — Let $T : \Omega \times X \to X$ be a continuous random operator such that

$$\langle T(\omega)x, x \rangle \geqslant 0 \text{ for } \| x \| = r.$$

Then there exists a random solution $x(\omega) \in B_r$ of the operator equation

$$T(\omega)\, x\omega = 0.$$

4. Random Equations in an Infinite Dimensional Space

Definition 4.1 — Let X be a Banach space, X is said to have an approximation scheme $\{X_n, P_n\}$ if there exists a sequence $\{X_n\}$ of finite dimensional subspaces of X and a sequence of projection operators P_n of X into X_n such that

$$X_1 \subset X_2 \subset \ldots \subset X_n \ldots, \overline{\bigcup_{n=1}} X_n = X, \; \| P_n \| \leqslant K.$$

Example 4.1 — Let X be a separable Banach space with countably linearly independent subset $\{x_n\}$ of X such that the linear span of $\{x_n\}$ is dense in X. Then the approximation scheme $\{X_n, P_n\}$ is defined as $X_1 = [x_1]$, $X_2 = [x_1, x_2]$, ..., $X_n = [x_1, x_2, \ldots x_n]$, ..., and $P_n x = \sum_{j=1}^{n} \alpha_j x_j$ where $x = \sum_{j=1}^{\infty} \alpha_j x_j$. Here $[x_1, x_2, \ldots, x_n]$ indicates linear span of $x_1, x_2, \ldots, x_n$.

Definition 4.2 — A nonlinear operator T is called P-compact with respect to a projection scheme $\{X_n, P_n\}$ if $P_n T$ is continuous in X_n for large n and if for any constant $p > 0$ and bounded sequence $\{x_n\}$ with $x_n \in X_n$ the sequence

$$\{g_n\} = \{P_n T x_n - p x_n\}$$

is strongly convergent, then there exists a strongly convergent subsequence $\{x_{n_k}\}$ and an element x in X such that $x_{n_k} \to x$ and $P_{n_k} T_{n_k} \to Tx$ as $k \to \infty$.

Remark 4.1 : If $p > 0$ the class of P-compact operators contains all closed precompact operators and all continuous, demi-continuous and weakly continuous operators T for which $-T$ is monotone on the Hilbert space H (refer Petryshyn 1966).

We have the following main theorem for P-compact random operators analogous to Theorem 2 of Petryshyn (1966) for deterministic operators.

Theorem 4.1 — Let X be a separable reflexive Banach space and $T: \Omega \times X \to X$ a P-compact bounded random operator. Suppose there exist constants $r > 0$ and $\mu(\omega) > 0$ [$\mu(\omega)$ is random] such that it satisfies the condition:

($\leqslant$) If for some x in S_r the operator equation $T(\omega)\, x = \alpha x$ holds, then $\alpha \leqslant \mu(\omega)$.

Then there exists a random solution $x(\omega)$ in $B_r - S_r$ of the equation

$$T(\omega)\, x(\omega) = \mu(\omega)\, x(\omega).$$

PROOF : Since X is a separable Banach space, as shown before, we have an approximation scheme $\{X_n, P_n\}$. By Lemma 1 of Petryshyn (1966), there exists an integer $N(\omega) > 0$ such that if $n \geqslant N(\omega)$ and $P_n T(\omega)\, x = \beta x$ for some x in $S_r \cap X_n$ then $\beta < \mu(\omega)$. The operator $P_n T(\omega)$, by definition, is a continuous operator of X_n into itself. Since $T(\omega)$ is random and P_n is continuous, it follows by Theorem 2.1 that the composition operator $P_n T(\omega)$ is random.

In view of the above stated lemma of Petryshyn, all the conditions of Theorem 3.1 are satisfied. Hence there exists a random variable $x_n(\omega)$ in $B_r \cap X_n$ such that

$P_n T(\omega)\, x_n(\omega) = \mu(\omega)\, x_n(\omega)$, $\| x_n(\omega) \| < r$ for all ω and for all $n \geqslant N(\omega)$.

Define a new sequence $u_n(\omega)$ as:

$$u_n(\omega) = \begin{cases} 0 & n < N(\omega) \\ x_n(\omega) & n \geqslant N(\omega). \end{cases}$$

Then $u_n(\omega)$ is measurable and further $\| u_n(\omega) \| \leqslant r$ for all $\omega \in \Omega$ and $n = 1, 2, \ldots$. Now we use the technique employed by Itoh (1978) to extract a measurable convergent subsequence out of $u_n(\omega)$. For each n, define $G : \Omega \to WK(B_r)$ by $G_n(\omega) =$ w.cl. $\{u_i(\omega) : i \geqslant n\}$ for each $\omega \in \Omega$. By Theorem 2.4 the mappings G_n are weakly measurable and so is the mapping $G : \Omega \to WK(B_r)$ defined by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$ (Theorem 2.3). It now follows, by Theorem 2.5, that there is a ω-measurable selector $x(\omega) : \Omega \to B_r$ of G. Because the space is separable it follows that $x(\omega)$ is also measurable [refer Hille and Phillips (1957)]. Since $x(\omega) \in G(\omega)$, it implies by the definition of $G(\omega)$ that there exists a subsequence $u_m(\omega)$ of $u_n(\omega)$ for which $u_m(\omega) \rightharpoonup x(\omega)$. Also $u_m(\omega) = x_m(\omega)$ for large m and $P_m\, T(\omega)\, x_m(\omega) - \mu(\omega)\, x_m(\omega) = 0$. P-compactness of $T(\omega)$ implies that there exists a subsequence $x_{m_j}(\omega)$ and an element

$x^*(\omega)$ such that $x_{m_j}(\omega) \to x^*(\omega)$ and $P_{m_j}T(\omega)\, x_{m_j}(\omega) \to T(\omega)\, x^*(\omega)$ and $T(\omega)\, x^*(\omega) - \mu(\omega)\, x(\omega) = 0$. Also $x_{m_j}(\omega) \rightharpoonup x(\omega)$, hence by the uniqueness of limits $x^*(\omega) = x(\omega)$. This implies that $T(\omega)\, x(\omega) = \mu(\omega)\, x(\omega)$. Since $x(\omega)$ is random, our theorem follows.

In view of Remark 4.1 we obtain the following for monotone random operators in a separable Hilbert space H.

Corollary 4.1 — Let $T : \Omega \times H \to H$ be a bounded, monotone, demicontinuous random operator. Suppose there exists $r > 0$ such that

$$\langle\, T(\omega)\, x, x \,\rangle \geqslant 0 \text{ for } \| x \| = r.$$

Then there exists a random solution $x(\omega)$ of the operator equation $T(\omega)\, x(\omega) = 0$.

Proof : As remarked before $-T(\omega)$ is a bounded P-compact random operator on H. We need only to verify the condition $(\leqslant)$. So let $-T(\omega)\, x = \alpha x$ for some x in S_r. Then $\langle -T(\omega)\, x, x \rangle = \alpha \langle x, x \rangle$. This implies that

$$\alpha = -\frac{\langle\, T(\omega)\, x, x \,\rangle}{\| x \|} \leqslant 0.$$

Hence by Theorem 4.1, there exists a random solution $x(\omega)$ such that $-T(\omega)\, x(\omega) = 0$. This in turn implies that $T(\omega)\, x(\omega) = 0$.

Remark 4.2 : The above corollary is a random analogue of the Browder's theorem (1964) for the deterministic monotone operator. The random analogue was first obtained by Itoh (1978). Author also obtains this by the method of first principle [refer Joshi (1979)].

Corollary 4.2 — If $T : \Omega \times X \to X$ is a P-compact, bounded random operator satisfying the condition $(\leqslant)$ for some $r > 0$, then $T(\omega)$ has a random fixed point $x(\omega)$ in $B_r - S_r$.

From the above corollary and Remark 4.1, we can now obtain the random analogues of well-known fixed point theorems of Schauder, Rothe, Altman and a comparison theorem in a separable reflexive Banach space X. We omit the proof as they are direct generalization of the corresponding known results for deterministic cases.

Corollary 4.3 — If $T(\omega)$ is a compact, continuous random operator of B_r into itself, then $T(\omega)$ has a random fixed point.

Remark 4.3 : The above theorem for random operators was first obtained by Bharucha-Reid (1976) but with an extra condition of compactness on the ball B_r.

Corollary 4.4 — If $T(\omega)$ is a compact continuous random operator mapping B_r into X such that $T(\omega)(S_r) \subset B_r$ for all $\omega \in \Omega$, then $T(\omega)$ has a random fixed point.

Corollary 4.5 — If $T(\omega)$ is a compact, continuous random operator of B_r into X such that $\| T(\omega)\, x - x \|^2 \geqslant \| T(\omega)\, x \|^2 - \| x \|^2$ for all $x \in S_r$, then $T(\omega)$ has a random fixed point in B_r.

Corollary 4.6 — Let $T(\omega)$ and $S(\omega)$ be two P-compact, bounded, continuous random operators of B_r into a Hilbert space H such that for all x in S_r,

$$\langle\, T(\omega)\, x, x \,\rangle \leqslant \| x \|^2 \text{ and } \| T(\omega)\, x - S(\omega)\, x \| \leqslant \| x - T(\omega)\, x \|.$$

Then $S(\omega)$ has a random fixed point in B_r.

5. Approximate Random Solutions with Applications

In this section we obtain the approximate solvability of the random operator equation

$$T(\omega)\, x(\omega) = \mu(\omega)\, x(\omega),\ x(\omega) \in B_r \qquad \text{...(5.1)}$$

in a separable reflexive Banach space X.

As we already know, since X is separable it has an approximation scheme $\{X_n, P_n\}$. We first define the approximate random solution.

Definition 5.1 — The approximate random solution $x_n(\omega)$ of eqn. (5.1) is the random solution of the equation

$$P_n T(\omega)\, x_n(\omega) = \mu(\omega)\, x_n(\omega). \qquad \text{...(5.2)}$$

In section 4 we discussed the existence of random solutions of (5.1). We would like to know under what condition the approximate random solution $x_n(\omega)$ converges to the solution $x(\omega)$ of (5.1).

Theorem 5.1 — Let the random operator $T : \Omega \times X \rightarrow X$ be as in Theorem 4.1. Then the approximate eqn. (5.2) has a random solution $x_n(\omega) \in B(\Omega, X, \mu)$ for large n; and for a fixed $\omega \in \Omega$ there exists a subsequence $x_{n_k}(\omega)$ of $x_n(\omega)$ converging pointwise to a solution of (5.1). If (5.1) has a unique deterministic solution $x(\omega)$ for a fixed $\omega \in \Omega$, the entire sequence $x_n(\omega)$ converges to $x(\omega)$ in $B(\Omega, X, \mu)$.

Proof : We have already proved in the previous section that (5.2) has a measurable solution $x_n(\omega) \in B_r \cap X_n$. Since $x_n(\omega)$ is measurable with

$$\int_{\Omega} \| x_n(\omega) \|\, d\mu < \infty,$$

$x_n(\omega)$ is also Bochner integrable. Existence of subsequence $x_{n_k}(\omega)$ of $x_n(\omega)$ has also been proved earlier.

For the last part we first note that since (5.1) has a unique deterministic solution $x(\omega)$ for a fixed $\omega \in \Omega$, it follows by Theorem 7 of Petryshyn (1966) that the entire sequence $x_n(\omega)$ converges to $x(\omega)$ pointwise. We now use the Lebesgue dominated convergence theorem to conclude that $\int_\Omega \| x_n(\omega) - x(\omega) \| \, d\mu \to 0$. This, by the definition of $B(\Omega, X, \mu)$, implies that $\| x_n - x \|_{B(\Omega, X, \mu)} \to 0$.

This proves our assertion.

For the application of the above theorem we consider a random Volterra integral equation of the type

$$x(t; \omega) = \int_0^t K(t, s, \omega) \, f(s, x(s); \omega) \, ds \qquad \ldots(5.3)$$

where $K(t, s, \omega)$ is a random kernel and $f(s, x, \omega)$ is a nonlinear random function. If we define random linear and nonlinear operators K and F as

$$[K(\omega)\, x]\,(t) = \int_0^t k(s, t; \omega)\, x(s)\, ds$$

$$[F(\omega)\, x]\,(t) = f(t, x(t); \omega)$$

then (5.3) is equivalent to the random operator equation

$$T(\omega)\, x(\omega) = x(\omega), \quad T(\omega) = K(\omega)\, F(\omega). \qquad \ldots(5.4)$$

In order that the operators $K(\omega)$ and $F(\omega)$, map the right space ($L^2\,[0, 1]$) into itself, we need to assume that

(1) $K(s, t; \omega) \in L^2\,[0, 1] \times L^2\,[0, 1]$ for all $\omega \in \Omega$

and

(2) $| f(s, x; \omega) | \leqslant [g(s) + h \mid x \mid]$ for all $\omega \in \Omega$, $g \in L^2\,[0, 1]$ and $b > 0$.

We set $X = L^2\,[0, 1]$. X has orthonormal basis $\{e_1, e_2, \ldots, e_n, \ldots\}$. As defined before we put $X_1 = [e_1]$, $X_2 = [e_1, e_2], \ldots, X_n = [e_1, e_2, \ldots, e_n], \ldots$. Projection operators $P_n : X \to X_n$ are defined as

$$P_n x = \sum_{j=1}^{n} \alpha_j e_j, \text{ where } x = \sum_{j=1}^{\infty} \alpha_j e_j.$$

Let $K = \sup_{\omega \in \Omega} \| k(s, t; \omega \|$ be finite. Then, if $bK < 1$, we can obtain a ball B_r which will be mapped into itself by $T(\omega)$. Since $T(\omega)$ is compact and continuous it follows by Corollary 4.3 that (5.3) has a random solution $x(\omega) \in B(\Omega, X, \mu)$. Further, if (5.3) has a unique solution for fixed $\omega \in \Omega$; it follows by the above theorem that the random approximate solutions $x_n(\omega)$ converge to $x(\omega)$ in

$$B(\Omega, L^2\,[0, 1], \mu).$$

REFERENCES

Bharucha-Reid, A. T. (1972). Random Integral Equations. Academic Press, New York.

———— (1976). Fixed point theorems in probabilistic analysis. *Bull. A.M.S.*, **82**, No. 5, 641-57.

Browder, F. E. (1964). Remarks on nonlinear functional equations. *Proc. natn. Acad. Sci., U.S.A.*, **51**, 985–89.

Hans, O. (1957). Reduzierende Zufallige Transformationen. *Czech. Math. J.*, 7(82), 154–58.

———— (1960). Random operator equations. *Proc. 4th Berkeley Symp. Math. Statistics and Prob.*, Vol. II.

Hille, Einar, and Phillips, Ralph H. (1957). Functional Analysis and Semigroups, Am. Math. Soc. Colloquium Publications, Vol. XXXI, New York.

Himmelberg, C. J. (1975). Measurable relations. *Fund. Math.*, **87**, 53–72.

Itoh, Shigeru (1978). Nonlinear random equation with monotone operators in Banach spaces. *Math. Ann.*, **236**, 133–46.

Joshi, Mohan (1979). Measurability of inverses of random operators and existence theorems. To appear in *Proc. Indian Acad. Sci.*

Kannan, R., and Salehi, H. (1976). Measurability of solutions of nonlinear equations. *Proc. Conf. Nonlinear Func. Anal., Michigan State Univ., Marcel. Dekker.*

Kuratowski, K., and Ryll-Nardzewski, C. (1965). A general theorem on selectors. *Bull. Acad. Polon. Sci. Ser. Math. Sci. Astr. Phys.*, **13**, 397–403.

Mukherjee, A. (1966). Transformation aleateires separables : Theoreme all point fixe aleateire. *C.R. Acad. Sci. Paris, Ser.* A-B **263**, A 393–A 395.

Nashed, M. S., and Salehi, H. (1973). Measurability of generalized inverses of random linear operators. *SIAM J. appl. Math.*, **25**, 681–92.

Petryshyn, W. V. (1966). On nonlinear *P*-compact operators in Banach space with applications to constructive fixed point theorem. *Math. Anal. Appl.*, **15**, 228–43.

Prakasa Rao, B. L. S. (1973). Stochastic integral equation of mixed type II. *J. Math. Phys. Sci.*, **7**, 245–60.

Spacek, A. (1955). Zufallige Gleichungen. *Czech. Math. J.*, **5**(80), 462–66.

Tsokos, C. P., and Padgett, V. J. (1974). Random Integral Equations with Applications to Life Sciences and Engineering. Academic Press, New York.