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SYMMETRY PROPERTIES OF CERTAIN GRAVITATIONAL SPACE-TIMES

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In this paper symmetry properties of the space-time formed from pure magnetic geons have been studied. It is found that the metric admits motion in general. It has also been established that the curvature collineation (CC) vector admitted by the space-time formed from zero-rest-mass scalar fields does not admit motion in general. The (CC) vector has been determined.

1. INTRODUCTION

The curvature collineation (CC) symmetry of a V_4, which admits a vector ξ^i, is defined by the condition

$$\mathcal{L}_\xi R^h_{ijk} = 0$$

which can be expressed in terms of partial derivatives as

$$\mathcal{L}_\xi R^h_{ijk} = R^h_{ijk,m}\,\xi^m - R^m_{ijk}\,\xi^h_{,m} + R^h_{mjk}\,\xi^m_{,i} + R^h_{imk}\,\xi^m_{,j} + R^h_{ijm}\,\xi^m_{,k} = 0. \qquad ...(1.1)$$

Here $\mathcal{L}_\xi$ stands for the Lie derivative with respect to the field vector ξ^i for the point transformation $\bar{x}^i = x^i + \xi^i(x)\,\delta t$, δt being a positive infinitesimal, and R^h_{ijk} the curvature tensor for the Riemannian space-time. Throughout this paper the comma and semicolon followed by any suffix, or suffixes, denote partial and covariant differentiation respectively with respect to the corresponding variables.

The search for curvature collineation and motion admitted by a space-time is essentially motivated by the study of conservation laws in relativistic framework. Katzin *et al.* (1969) have investigated various properties of geometrical and physical interest associated with (CC) vectors, and have established a number of theorems connecting various symmetry properties with curvature collineation. An important feature of the (CC) is that it indicates a type of elastic deformation of the space-time for which the gravitational properties are preserved along the curvature collineation vector. We call a (CC) vector Killing if it defines motion in the chain of higher symmetries such as motion, homothetic motion, conformal motion, affine collineation, projective collineation, etc. A (CC) is called proper if it does not admit any of the higher symmetries like those mentioned above. Recenty, Singh *et al.* (1978)

considered the cylindrically symmetric field of total radiation obtained by Rao (1964) for (CC) and found that it admits proper curvature collineation.

In section 2 of this paper we have considered Melvin's magnetic universe for symmetry properties. Here it is found that the collineation vector is Killing. Section 3 is devoted to the study of space-time corresponding to zero-rest-mass scalar fields. It has been shown that the metric admits proper curvature collineation in general.

2. Space-time Corresponding to Pure Magnetic Geons

Melvin (1964) has given an exact axisymmetric solution of the combined sourceless Einstein-Maxwell field equations in static case. The solution represents a parallel bundle of magnetic flux held together by its own gravitational field with the persistent local energy-stress concentration which may be taken as the defining characteristic of geon. Melvin's magnetic universe is described by the metric

$$ds^2 = e^{2\psi}(dt^2 - dr^2 - dz^2) - r^2e^{-2\psi}\,d\phi^2 \qquad \ldots(2.1)$$

where ψ is the function of r alone and the cylindrical polar co-ordinates r, ϕ, z, t correspond to x^1, x^2, x^3, x^4 respectively.

For the metric (2.1), the non-vanishing components of the Riemann curvature tensor R^h_{ijk} are

$$\left.\begin{aligned}
&R^1_{331} = R^1_{414} = R^3_{113} = R^4_{114} = \psi'',\\
&R^3_{232} = R^4_{242} = r^2e^{-4\psi}R^2_{323} = r^2e^{-4\psi}R^2_{442} = r^2e^{-4\psi}(\psi'^2 - \psi'/r),\\
&R^1_{212} = r^2e^{-4\psi}R^2_{121} = r^2e^{-4\psi}(\psi'' - 2\psi'^2 + 3\psi'/r),\\
&R^3_{434} = R^4_{334} = \psi'^2
\end{aligned}\right\} \qquad \ldots(2.2)$$

where ($'$) denotes ordinary differentiation with respect to r.

From the algebraic symmetries on the indices we find that eqn. (1.1) formally represents 96 equations. Considering these equations for the line element (2.1), we obtain the following set of equations for the non-vanishing and vanishing components of the Riemann curvature tensor:

$$\mathcal{L}_\xi R^1_{212} = 0 \Rightarrow \{r^2e^{-4\psi}(\psi'' - 2\psi'^2 + 3\psi'/r)\}_{,1}\,\xi^1$$

$$+ 2r^2e^{-4\psi}(\psi'' - 2\psi'^2 + 3\psi'/r)\,\xi^2_{,2} = 0 \qquad \ldots(2.3)$$

$$\mathcal{L}_\xi R^3_{223} = 0 \Rightarrow \{r^2e^{-4\psi}(\psi'^2 - \psi'/r)\}_{,1}\,\xi^1 + 2r^2e^{-4\psi}(\psi'^2 - \psi'/r)\,\xi^2_{,2} = 0 \qquad \ldots(2.4)$$

$$\mathcal{L}_\xi R^1_{313} = 0 \Rightarrow (\psi'')_{,1}\, \xi^1 + 2\psi''\xi^3_{,3} = 0 \qquad \text{...(2.5)}$$

$$\mathcal{L}_\xi R^4_{334} = 0 \Rightarrow (\psi'^2)_{,1}\, \xi^1 + 2\psi'^2\xi^3_{,3} = 0 \qquad \text{...(2.6)}$$

$$\mathcal{L}_\xi R^2_{323} = 0 \Rightarrow (\psi'^2 - \psi'/r)_{,1}\, \xi^1 + 2(\psi'^2 - \psi'/r)\, \xi^3_{,3} = 0 \qquad \text{...(2.7)}$$

$$\mathcal{L}_\xi R^1_{414} = 0 \Rightarrow (\psi'')_{,1}\, \xi^1 + 2\psi''\xi^4_{,4} = 0 \qquad \text{...(2.8)}$$

$$\mathcal{L}_\xi R^3_{434} = 0 \Rightarrow (\psi'^2)_{,1}\, \xi^1 + 2\psi'^2\xi^4_{,4} = 0 \qquad \text{...(2.9)}$$

$$\mathcal{L}_\xi R^2_{424} = 0 \Rightarrow (\psi'^2 - \psi'/r)_{,1}\, \xi^1 + 2(\psi'^2 - \psi'/r)\, \xi^4_{,4} = 0 \qquad \text{...(2.10)}$$

$$\mathcal{L}_\xi R^3_{113} = 0 \Rightarrow (\psi'')_{,1}\, \xi^1 + 2\psi''\xi^1_{,1} = 0 \qquad \text{...(2.11)}$$

$$\mathcal{L}_\xi R^2_{112} = 0 \Rightarrow (\psi'' - 2\psi'^2 + 3\psi'/r)_{,1}\, \xi^1 + 2(\psi'' - 2\psi'^2 + 3\psi'/r)\, \xi^1_{,1} = 0 \qquad \text{...(2.12)}$$

$$\mathcal{L}_\xi R^1_{112} = 0 \Rightarrow r^2 e^{-4\psi}\xi^2_{,1} + \xi^1_{,2} = 0 \qquad \text{...(2.13)}$$

$$\mathcal{L}_\xi R^2_{22i} = 0 \Rightarrow r^2 e^{-4\psi}\xi^2_{,i} \pm \xi^i_{,2} = 0, \quad i = 3, 4 \qquad \text{...(2.14)}$$

$$\mathcal{L}_\xi R^1_{314} = 0 \Rightarrow \xi^3_{,4} - \xi^4_{,3} = 0 \qquad \text{...(2.15)}$$

$$\mathcal{L}_\xi R^1_{11i} = 0 \Rightarrow \xi^1_{,i} \pm \xi^i_{,1} = 0, \quad i = 3, 4 \qquad \text{...(2.16)}$$

$$\mathcal{L}_\xi R^1_{323} = 0 \Rightarrow \xi^1_{,2} = 0 \qquad \text{...(2.17)}$$

$$\mathcal{L}_\xi R^1_{22i} = 0 \Rightarrow \xi^1_{,i} = 0, \quad i = 3, 4 \qquad \text{...(2.18)}$$

$$\mathcal{L}_\xi R^2_{11i} = 0 \Rightarrow \xi^2_{,i} = 0. \qquad \text{...(2.19)}$$

Here the trivial equations have been omitted.

On imposing the condition $\xi^1 = 0$ for the (CC) vector ξ^i, the set of eqns. (2.3) – (2.19) gives the following solution for ξ^i:

$$\left.\begin{aligned} &\xi^1 = 0, \ \xi^2 = \alpha \\ &\xi^3 = \beta t + \gamma, \quad \xi^4 = \beta z + \delta \end{aligned}\right\} \qquad \text{...(2.20)}$$

where α, β, γ and δ are arbitrary constants.

Now the eqns. (2.3) – (2.19) are satisfied for the values of ξ^i given above. Hence the space-time (2.1) corresponding to Melvin's magnetic universe admits the (CC) vector ξ^i given by (2.20).

In order to decide whether the (CC) vector ξ^i is a Killing vector, we proceed as follows. We know that the space-time admits motion iff

$$h_{ij} \equiv \mathcal{L}_\xi g_{ij} = \xi_{i;j} + \xi_{j;i} = 0. \qquad \text{...(2.21)}$$

It follows from (2.20) that eqn. (2.21) is identically satisfied. Hence the (cc) vector ξ^i admitted by Melvin's geon space-time defines a Killing vector.

3. Space-Time Corresponding to Zero-Rest-Mass Scalar Fields

Recently, Singh *et al.* (1979) have found an exact solution of Einstein's field equations for zero-rest-mass scalar fields corresponding to a stationary cylindrically symmetric space-time. The solution is given by the axisymmetric static metric

$$ds^2 = dt^2 - r^b(dr^2 + dz^2) - r^2\, d\phi^2 \qquad \text{...(3.1)}$$

where b is a non-zero constant characterising the scalar field.

For the line element (3.1), the non-zero components of the Riemann curvature tensor $R^h_{\ ijk}$ are given by

$$R^1_{\ 313} = -R^2_{\ 323} = r^{b-2} R^1_{\ 212} = \frac{b}{2r^2}. \qquad \text{...(3.2)}$$

Now the eqn. (1.1), for the metric (3.1), gives the following set of equations for the surviving and non-surviving components of the Riemann curvature tensor:

$$\mathcal{L}_\xi R^1_{\ 212} = 0 \Rightarrow \frac{b}{2r}\,\xi^1 = \xi^2_{\ ,2} \qquad \text{...(3.3)}$$

$$\mathcal{L}_\xi R^1_{\ 313} = 0 \Rightarrow \xi^1 = r\xi^3_{\ ,3} \qquad \text{...(3.4)}$$

$$\mathcal{L}_\xi R^1_{\ 213} = 0 \Rightarrow \xi^3_{\ ,2} + r^{2-b}\xi^2_{\ ,3} = 0 \qquad \text{...(3.5)}$$

$$\mathcal{L}_\xi R^1_{\ 11i} = 0 \Rightarrow \xi^i_{\ ,1} = 0, \quad i = 2, 3 \qquad \text{...(3.6)}$$

$$\mathcal{L}_\xi R^1_{\ i14} = 0 \Rightarrow \xi^i_{\ ,4} = 0, \quad i = 2, 3 \qquad \text{...(3.7)}$$

$$\mathcal{L}_\xi R^1_{\ 22i} = 0 \Rightarrow \xi^1_{\ ,i} = 0, \quad i = 3, 4 \qquad \text{...(3.8)}$$

$$\mathcal{L}_\xi R^i_{\ 323} = 0 \Rightarrow \xi^i_{\ ,2} = 0, \quad i = 1, 4 \qquad \text{...(3.9)}$$

$$\mathcal{L}_\xi R^2_{\ 223} = 0 \Rightarrow \xi^3_{\ ,2} = 0 \qquad \text{...(3.10)}$$

$$\mathcal{L}_\xi R^4_{\ 212} = 0 \Rightarrow \xi^4_{\ ,1} = 0 \qquad \text{...(3.11)}$$

The redundant and trivial equations have been omitted. By inspection we find that the eqns. (3.3) – (3.11) lead to the following solution for ξ^i :

$$\left.\begin{aligned} \xi^1 &= \frac{2A}{b}\, r,\ \xi^2 = A\phi + B, \\ \xi^3 &= \frac{2A}{b}\, z + C,\ \xi^4 = A(z + t) + D \end{aligned}\right\} \qquad \ldots(3.12)$$

where A, B, C and D are arbitrary constants. Now the eqns. (3.3) – (3.11) are satisfied for the values of ξ^i as given here in (3.12). Hence the space-time (3.1) formed from zero-rest-mass scalar fields admits the collineation vector ξ^i given by (3.12).

Now we shall see that the (CC) vector ξ^i admitted by the metric (3.1) is proper. By the use of (3.12) and the definition $h_{ij} \equiv \mathcal{L}_\xi g_{ij} = \xi_{i;j} + \xi_{j;i}$, we see that $h_{ij} \neq 0$ unless $A = 0$; which shows that the (CC) vector ξ^i does not define motion in general.

For affine collineation ξ^i must satisfy $\mathcal{L}_\xi \Gamma^k_{ij} = 0$. It is found that

$$\mathcal{L}_\xi \Gamma^4_{13} = -\frac{Ab}{2r} \neq 0.$$

Therefore ξ^i is not an affine collineation vector. It can easily be seen that $\xi^i_{;ijk} \neq 0$ in general. Thus ξ^i neither defines a conformal motion (including homothetic motion) nor projective and conformal collineations. Hence the (CC) vector ξ^i admitted by the space-time (3.1) is proper. In general this is a four-parameter group of transformations defining a proper (CC). In particular when $A = 0$ it reduces to a three-parameter group of motions.

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