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A GENERALIZATION OF THE NEWTON-CARTAN THEORY OF GRAVITATION

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It is shown that even in the absence of the equivalence principle, the Newtonian theory of gravitation can be given a geometric form in a five-dimensional manifold. The fifth dimension is taken as the ratio of gravitational and inertial mass, which we allow to be different for different particles. The resulting pondoromotive and field equations in this 5-dimensional space (which are generalizations of Cartan's formulation of Newtonian gravitation) are formulated and their consequences are discussed. It is argued that as general relativity is a 'metric' theory, a similar generalization of general relativity is not possible.

1. INTRODUCTION

In the classical formulation of Newtonian gravitation, the constancy of the ratio of gravitational mass and inertial mass for all particles (i.e. the 'equivalence principle') is not a basic axiom or a theoretical deduction, but just an accidental emperical fact. However, in Cartan's differential geometric formulation of Newtonian gravitation, the equivalence principle is taken as a basic axiom. Note that an auto-parallel curve in an affine space is uniquely determined by a point on it and the tangent to it at that point, and the same is true for the world lines of freely falling particales when the equivalence principle is valid. In Cartan's theory, the equivalence principle is used to define an affine connection on space-time, of which paths of freely falling particles are autoparallel curves.

In this note, we have generalized Cartan's theory by dropping the equivalence principle from the list of axioms. Instead, we assume that the ratio of gravitational mass to inertial mass can assume various but fixed values for various particles. These values are assumed to lie in some open interval I on the real line. To preserve the property of Cartan's theory that freely falling particles follow autoparallel curves, the space is taken to be 5-dimensional : it is assumed to be the product of the 4-dimensional space-time with the interval I. Two particles having the same position in the 4-space-time but different values of the ratio gravitational mass/inertial mass have different positions in this 5-space, and so the path of a freely falling particle in the 5-space is uniquely determined by its position (in the 5-space) and the tangent to the path at that position. We show that an affine connection can be defined on the 5-space of which the paths of freely falling particles are autoparallel curves.

In section 2, we have given a resume of the Newton-Cartan theory for ready comparison. In section 3, we develop the pondoromotive and field equations of Newtonian gravitation in our 5-space and discuss a particular consequence of these that the concept of an 'inertial' frame is stronger in our 5-space than in the 4-space-time of Cartan. In section 4, we give a coordinate-free geometric formulation of our generalization of Cartan's theory and we establish the equivalence of this coordinate-free formulation with the formulation of section 3. In section 5, we argue that a similar 5-dim. generalization of general relativity is not possible as general relativity is a 'metric' theory. This also applies to other 'metric' theories of gravitation. We conclude our arguments in section 6.

Notation and Conventions

The usual summation convention is used for greek indices α, β, γ etc. which run from 1 to 4 in section 2, and from 1 to 5 in section 3-5 unless otherwise stated. The latin indices i, j etc. run from 1 to 3, and the 'Euclidean' summation convention [i.e., a repeated index is to be summed over irrespective of its position (super/sub)] is employed for them. In all other respects, we have tried to follow the notation of Misner *et al.* (1973).

2. A Resume of the Newton-Cartan Theory

Here we have summarized the laws of the Newton-Cartan theory for ready reference and comparison, following Misner *et al.* (1973). The axioms of the theory are as follows:

(1) There exists a universal time t, a set of Cartesian space coordinates X^j (called Galilean coordinates), and a Newtonian gravitational potential ϕ.

(2) The density of mass ρ generates the Newtonian potential by Poisson's equation,

$$\frac{\partial^2 \phi}{\partial X^j \partial X^j} = 4\pi\rho. \qquad ...(1)$$

(3) The equation of motion for a freely falling particle is

$$\frac{d^2 X^j}{dt^2} + \frac{\partial \phi}{\partial X^j} = 0. \qquad ...(2)$$

(4) "Ideal rods" measure the Galilean coordinates lengths; "ideal clocks" measure universal time.

From these assumptions, it is derived that freely falling particles follow the autoparallel curves

$$\frac{d^2 X^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dX^\beta}{d\lambda} \frac{dX^\gamma}{d\lambda} = 0 \qquad ...(3)$$

where the affine parameter λ is linearly related to the universal time t. In terms of a Galilean coordinate system, the coefficients of the affine connection are

$$\Gamma^i_{44} = \frac{\partial \phi}{\partial X^i} \qquad ...(4)$$

while rest of the components are zero. The curvature tensor $R^{\alpha}_{\beta\gamma\delta}$ in terms of a Galilean coordinate system is

$$R^i_{4j4} = -R^i_{44j} = \frac{\partial^2 \phi}{\partial X^i \partial X^j} \qquad ...(5)$$

while rest of the components are zero. So the field equations can be expressed in terms of the Ricci tensor in Galilean coordinates as

$$R_{44} = 4\pi\rho \qquad ...(6)$$

Two different Galilean coordinate systems (X^1, X^2, X^3, X^4) and $(X^{1'}, X^{2'}, X^{3'}, X^{4'})$ are connected by the equations

$$X^{4'} = \pm X^4 + \text{constant} \qquad ...(7)$$

$$X^{j'} = A_{j'k} X^k + a^{j'} \qquad ...(8)$$

where $A_{j'k}$ and $a^{j'}$ are functions of t, $A_{j'k}$ is a 3×3 orthogonal matrix, and

$$\dot{A}_{j'k} = 0 \qquad ...(9)$$

(where the dot denotes time derivative). Note that $\ddot{a}^{j'}$ is not required to be zero, so two Galilean frames can be accelerated w.r.t. each other.

The Newtonian potentials ϕ and ϕ' in the two systems are related by

$$\phi' = \phi - \ddot{a}^k X^k + \text{constant}. \qquad ...(10)$$

3. The Generalization

We base our generalization on the following axioms, which are generalizations of the corresponding axioms for the Newton-Cartan theory:

(1) There exists a universal time t, a set of Cartesian space coordinates X^j (called Galilean coordinates), and a Newtonian gravitational potential ϕ.

(2) The density of gravitational mass ρ generates the Newtonian potential by Poisson's equation,

$$\frac{\partial^2 \phi}{\partial X^j \partial X^j} = 4\pi\rho. \qquad ...(11)$$

(3) The equation of motion for a freely falling particle with gravitational mass m_G and inertial mass m_I is

$$m_I \frac{d^2X^j}{dt^2} + m_G \frac{\partial\phi}{\partial X^j} = 0. \qquad \text{...(12)}$$

The value of m_I for any particle is non-zero. The ratio m_G/m_I is fixed for a given particle, but for different particles, the ratio can take any value in some open interval I on the real line.

(4) "Ideal rods" measure the Galilean coordinate lengths, "Ideal clocks" measure the universal time.

Now let us introduce the new dimension

$$X^5 = \frac{m_G}{m_I}. \qquad \text{...(13)}$$

Then eqn. (12) can be re-written as

$$\frac{d^2X^j}{dt^2} + X^5 \frac{\partial\phi}{\partial X^j} = 0. \qquad \text{...(14)}$$

Let $\lambda = at + b$ where a and b are constants. Then from (14),

$$\frac{d^2X^j}{d\lambda^2} + X^5 \frac{\partial\phi}{\partial X^j} \left(\frac{dt}{d\lambda}\right)^2 = 0. \qquad \text{...(15)}$$

Taking $X^4 = \pm\, t +$ constant, (15) gives

$$\frac{d^2X^j}{d\lambda^2} + X^5 \frac{\partial\phi}{\partial X^j} \frac{dX^4}{d\lambda} \frac{dX^4}{d\lambda} = 0. \qquad \text{...(16)}$$

Comparing (16) with eqn. (3) for autoparallel curves, we get

$$\Gamma^i_{44} = X^5 \frac{\partial\phi}{\partial X^i} \qquad \text{...(17)}$$

while rest of the coefficients vanish. So, the paths of freely falling particles can be considered as the autoparallel curves of the affine connection given by (17), and the canonical parameter λ is linearly related to the universal time.

From (17), we can calculate the expression for the curvature tensor in the Galilean coordinates. We obtain

$$R^i_{4j4} = -R^i_{44j} = X^5 \frac{\partial^2\phi|}{\partial X^i \partial X^j} \qquad \text{...(18)}$$

and

$$R^i_{454} = -R^i_{445} = \frac{\partial\phi}{\partial X^i} \qquad \text{...(19)}$$

while rest of the components are zero.

From (18), the Ricci tensor is

$$R_{44} = X^5\, 4\pi\rho. \qquad \text{...(20)}$$

(Note that we regard the density of gravitational mass ρ to be independent of X^5). These are the fundamental equations for our 5-dimensional theory.

Note that when $X^5 = 1$, the eqn. (14), (17), (18) and (20) reduce to the corresponding eqns. (2), (4), (5) and (6) of the Newton-Cartan theory. So the Newton-Cartan spacetime can be identified with the hypersurface $X^5 = 1$ (i.e. gravitational mass = inertial mass) of our 5-dim. space. We shall call it as the Newton-Cartan hypersurface.

When $X^5 = 0$, it is easy to see that we obtain the 'gravitation-free' Galilean space-time. Now we shall try to interpret the term R^i_{454} of the curvature, given by eqn. (19). Note that this term is independent of X^5. We shall show that because of the presence of this term R^i_{454} given by (19), the class of all Galilean frames is more restricted than in the Newton-Cartan space-time, and now the transformations connecting two Galilean frames are precisely the Galilean transformations given by

$$X^{5'} = X^5 \qquad \text{...(21)}$$

$$X^{4'} = \pm X^4 + \text{constant} \qquad \text{...(22)}$$

$$X^{j'} = A_{j'k}X^k + V^{j'}X^4 + C^{j'} \qquad \text{...(23)}$$

where $A_{j'k}$ is a constant 3×3 orthogonal matrix, and $V^{j'}$ and $C^{j'}$ are constants. Also, we shall show that the Newtonian potentials, ϕ' and ϕ are related by

$$\phi' = \phi + \text{constant}. \qquad \text{...(24)}$$

We now establish these equations. Equations (21) and (22) are obvious. To prove (23), we observe that as both $(X^1, \ldots, X^5)$ and $(X^{1'}, \ldots, X^{5'})$ are Galilean sets of coordinates, (X^1, X^2, X^3) and $(X^{1'}, X^{2'}, X^{3'})$ are both Cartesian coordinates in a 3-dim. Euclidean space, so are related by

$$X^{j'} = A_{j'k}X^k + a^{j'} \qquad \text{...(25)}$$

where $A_{j'k}$ is an orthogonal matrix, and both $A_{j'k}$ and $a^{j'}$ are functions of time.

Now, in the first coordinate system $\Gamma^\alpha_{\beta\gamma}$ are given by eqn. (17). So, calculating $\Gamma^{\alpha'}_{\beta'\gamma'}$ we get

$$\Gamma^{j'}_{4'k'} = \Gamma^{j'}_{k'4'} = \pm A_{j'l}\dot{A}_{k'l} \qquad \text{...(26)}$$

$$\Gamma^{j'}_{4'4'} = X^{5'} \frac{\partial \phi}{\partial X^{j'}} + A_{j'k}(\ddot{A}_{l'k}X^{l'} - \ddot{a}^{k'}) \qquad \text{...(27)}$$

where dot denotes derivative with time X^4 and a^k is given by $a^k = A_{j'k} a^{j'}$, that is, $X^k = A_{j'k} X^{j'} - a^k$. Hence, $\Gamma^{\alpha'}_{\beta'\gamma'}$ satisfy the standard form (17) if and only if

$$\dot{A}_{k'l} = 0 \qquad \text{...(28)}$$

and $$X^{5'}\phi' = X^5\phi - \ddot{a}^k X^k + \text{constant.} \qquad \text{...(29)}$$

Because of (28), only the $a^{i'}$ in (25) are functions of time, while $A_{j'k}$ are constant.

Now, calculating the component $R^{i'}_{4'5'4'}$ of the curvature, we get

$$R^{i'}_{4'5'4'} = \frac{\partial\phi}{\partial X^{i'}}. \qquad \text{...(30)}$$

Comparing this with the standard form (19), we get

$$\frac{\partial\phi'}{\partial X^{i'}} = \frac{\partial\phi}{\partial X^{i'}}. \qquad \text{...(31)}$$

Hence, $\phi' = \phi + \text{constant}$, so eqn. (24) is now proved. Comparing (24) with (29), we obtain the condition

$$\ddot{a}^k = 0 \qquad \text{...(32)}$$

as $a^{i'} = A_{i'k} a^k$, this gives

$$a^{i'} = V^{i'} X^4 + C^{i'} \qquad \text{...(33)}$$

where $V^{i'}$ and $C^{i'}$ are constants. This establishes (23) and so we have established eqns. (21) – (24). Note that the group of transformations given by (22) and (23) is a proper sub-group of the group of transformations given (7) and (8). Also, according to (23), two Galilean frames move with a constant relative velocity. This conclusion is obvious even on physical consideration: if particles with different ratios of m_G/m_I were available, Einstein's 'elevator experiments' would not work. Indeed, the extra term R^i_{454} given by eqn. (19), gives the geodesic deviation between the paths of two stationary particles having the same initial position in the space-time, but different values of m_G/m_I, consequently, different positions in our 5-dim. space.

4. A Coordinate-free Geometric Formulation

From our formulation of the 5-dim. theory in terms of Galilean coordinates in section 3, we can see that the coordinate-free geometric axioms for the theory (which are given below) are satisfied. The curve C of the axiom 12 can be identified with the curve given by $X^1 = X^2 = X^3 = 0$, $X^5 = 1$ in terms of Galilean coordinates. Conversely, from these coordinate-free geometric axioms, the standard axioms of section 3 are deduced at the end of this section.

Axiom 1 — The space S_5 is the product of the manifolds R^4 and I where I is an open interval (not necessarily bounded).

$$S_5 = \mathbf{R}^4 \times I. \qquad ...(34)$$

Axiom 2 — There is a symmetric covariant derivative ∇ on S_5.

Definition — An 'event-curve' is a curve in a S_5 of the form $\{a\} \times I$ where $a \in \mathbf{R}^4$. The nth space-time $S_4(n)$, where $n \in I$, is the hypersurface

$$S_4(n) = \mathbf{R}^4 \times \{n\} \qquad ...(35)$$

of S_5. Note that $n \in I$ is a parameter for all event-curves. The corresponding field of tangent vectors $\frac{d}{dn}$ will be denoted by N.

Axiom 3 — The 1-form field dn is covariantly constant, i.e.,

$$\nabla dn = 0. \qquad ...(36)$$

Axiom 4 — The field of tangent vectors N to the event curves is covariantly constant, i.e.,

$$\nabla N = 0. \qquad ...(37)$$

Axiom 5 — There is a real-valued function t, called 'universal time' defined on S_5 such that

$$\text{(i)} \quad dt \neq 0, \qquad \text{(ii)} \quad \nabla dt = 0, \qquad \text{(iii)} \quad \langle dt, N \rangle = 0. \qquad ...(38)$$

Definition — A vector U is called 'timeless' if $\langle dt, U \rangle = 0$. A vector U is called 'spatial' if $\langle dt, U \rangle = \langle dn, U \rangle = 0$.

Lemma 1 — If U is a spatial vector field, then $\nabla_p U$ is also spatial for any vector p.

Axiom 6 — If a vector u is timeless, then

$$\mathcal{R}(p, q)\, u = 0 \qquad ...(39)$$

where p and q are any vectors.

($\mathcal{R}$ is used to denote the Riemannian curvature operator).

Axiom 7 — If vectors u and v are timeless, then

$$\mathcal{R}(u, v) = 0. \qquad ...(40)$$

Axiom 8 — There is a real-valued function ρ, called 'density of gravitational mass', defined on S_5 such that

$$\langle d\rho, N \rangle = 0. \qquad ...(41)$$

Axiom 9 — The Ricci tensor is given by

$$\text{Ricci} = 4\pi n\rho \, dt \otimes dt. \qquad \ldots(42)$$

Axiom 10 — There is a positive definite metric, denoted by "." for spatial vectors only, such that for spatial vector fields u and v and for any vector p,

$$\nabla_p(u \cdot v) = (\nabla_p u) \cdot v + u \cdot (\nabla_p v). \qquad \ldots(43)$$

(Note that from Lemma 1, $\nabla_p u$ and $\nabla_p v$ are spatial).

Lemma 2 — For any vectors u, p, q, $\mathcal{J}(p, q)\, u$ is a spatial vector, where $\mathcal{J}$ is used to denote the Jacobi curvature operator.

Axiom 11 — For any spatial vectors u, v and for any arbitrary vectors p, q,

$$u \cdot [\mathcal{J}(p, q)\, v] = v \cdot [\mathcal{J}(p, q)\, u]. \qquad \ldots(44)$$

Axiom 12 — There is a curve C in S_5 having t as a parameter. Let the tangent vectors $\frac{d}{dt}$ to C be denoted by A. Then A satisfies the condition $\langle dn, A \rangle = 0$ at all points of C.

Lemma 3 — Let the tangent vector A at each point of C be parallel-transported throughout its hypersurface t = constant by arbitrary routes lying in each hypersurface to obtain a vector field A defined all over S_5. The resultant field A is independent of the routes of parallel transport in each hypersurface, i.e., A is 'well-defined'.

Axiom 13 — For the vector field A defined above,

$$\nabla_N(\mathcal{R}(N, A)\, A) = 0. \qquad \ldots(45)$$

Axiom 14 — $\nabla_A A = n\, \mathcal{R}(N, A)\, A$. $\qquad \ldots(46)$

Axiom 15 — There exist 'ideal clocks' to measure t, 'ideal rods' to measure the legth calculated with the spatial metric, and 'ideal ratio-finders' to measure $n = m_G/m_I$ for all particles. Freely falling particles follow the autoparallel curves of ∇.

Theorem — If the above axioms are satisfied, then there exists a coordinate system X^α for S_5 such that

(i) $e_5 \equiv \dfrac{\partial}{\partial X^5} = N$;

(ii) $X^5 = n$, $X^4 = \pm\, t$ + constant;

(iii) X^1, X^2, X^3 are rectangular Cartesian coordinates for spatial vectors;

(iv) there is a function ϕ (the Newtonian potential) on S_5 such that $\phi_{,5} = 0$;

(v) the equations of motion for freely falling particles are

$$\frac{d^2X^i}{dt^2} + n\frac{\partial\phi}{\partial X^i} = 0;$$

(vi) The potential ϕ satisfies the equation

$$\phi_{,ii} = 4\pi\rho.$$

PROOFS : Lemma 1 follows immediately from the axioms $\nabla dn = \nabla dt = 0$ and the definition of a spatial vector. Similarly, Lemma 3 follows immediately from axiom 7.

PROOF OF LEMMA 2 — At some point P in S_5, choose any three orthonormal spatial vectors e_1, e_2, e_3. ($e_i \cdot e_k = \delta_{ik}$) Parallel transport them throughout S_5 by arbitrary routes. From axiom 6, the resultant vector fields e_j are independent of the transport routes. Note that for the fields e_j, $\nabla e_j = 0$ and $[e_i, e_k] = 0$. Now, let $e_4 = A$ and $e_5 = N$. Then from the construction of the vector fields e_j and A, and the axiom $\nabla N = 0$, we see that $[e_\alpha, e_\beta] = 0$ for all α, β. Also $e_j \cdot e_k = \delta_{jk}$ everywhere. From the construction of A, $\langle dn, A \rangle = 0$, $\langle dt, A \rangle = 1$ everywhere. From all this, $e_1, \ldots, e_5$ is a coordinate basis, with $e_\alpha = \frac{\partial}{\partial X^\alpha}$, such that X^k are rectangular Cartesian coordinates for spatial vectors, $X^4 = t +$ constant and $X^5 = n +$ constant. We can redefine X^5 now such that $X^5 = n$. So now the first three conditions of the theorem are satisfied. From the conditions

$$\nabla dn = \nabla dt = \nabla e_5 = \nabla e_j = 0,$$

we can see that the only components $\Gamma^\alpha_{\beta\gamma}$ that may be non-zero are Γ^j_{44}. From this and the expression for the Jacobi curvature tensor in terms of $\Gamma^\alpha_{\beta\gamma}$, Lemma 2 follows immediately.

Proof of the theorem — We have already seen that the coordinate system X^α satisfies the first three conditions of the theorem. When in the statement of axiom 11 we put

$$u = e_j,\ v = e_k,\ p = q = e_4$$

and use the expression for the Jacobi curvature operator in terms of the Riemannian curvature operator, we obtain

$$R^j_{4k4} = R^k_{4j4}. \qquad \ldots(47)$$

As all $\Gamma^\alpha_{\beta\gamma}$ except the Γ^j_{44} are zero, this means

$$\Gamma^j_{44,k} = \Gamma^k_{44,j}. \qquad \ldots(48)$$

Hence there exists a scalar function ψ on S_5 such that

$$\Gamma^{i}_{44} = \psi_{,i}. \qquad \ldots(49)$$

The axiom 13 now reads

$$\frac{\partial}{\partial X^5} R^{i}_{454} = 0. \qquad \ldots(50)$$

In terms of ψ,

$$R^{i}_{454} = \psi_{,i5}. \qquad \ldots(51)$$

The axiom 14 reads

$$\Gamma^{i}_{44} = nR^{i}_{454}. \qquad \ldots(52)$$

But as $\frac{\partial}{\partial X^5} R^{i}_{454} = 0$ we can write

$$\psi_{,i} = X^5\phi_{,i} \qquad \ldots(53)$$

where $\phi_{,i}$ is independent of X^5. Hence,

$$R^{i}_{454} = \phi_{,i}. \qquad \ldots(54)$$

Now, we shall redefine ϕ such that $\phi_{,5} = 0$. For this, choose any fixed non-zero $a \in I$ and extend the values of ϕ on $S_4(a)$ to all of S_5 by keeping them constant along each event curve ('parallel transport of a scalar'!). As this does not affect the values of $\phi_{,i}$, the $\Gamma^{\alpha}_{\beta\gamma}$ are the same as before, but now $\phi_{,5} = 0$. It is now easy to see that this redefined ϕ satisfies the last two conditions of the theorem, and so the theorem is proved.

5. Generalization of General Relativity

Now we shall show that as general relativity is a 'metric' theory in which light follows null geodesics, a 5-dim. generalization of it on the lines of section 4 is not possible. Our considerations below will also be applicable to other 'metric' theories of gravitation in which freely falling bodies follow geodesics and in particular, light follows null geodesics.

We shall use, without giving a proof, the following elementary result from linear algebra:

Lemma 1 — Let V be a finite dimensional linear space over $\mathbf{R}$. Let g be a non-degenerate, indefinite, symmetric bilinear form on V. Let h be another symmetric bilinear form on V such that

$$\forall\, X \in V,\ g(X, X) = 0 \Rightarrow h(X, X) = 0.$$

Then h is a multiple of g, i.e.,

$$h = ag \text{ where } a \in \mathbf{R}.$$

(In particular, if h is non-degenerate, then $a \neq 0$).

Now assume that a generalization of general relativity somewhat on the lines of section 4, were possible. Then we would expect the generalized 5-dim. theory to satisfy the following axioms:—

Axiom 1 — $S_5 = \mathbf{R}^4 \times I$ (as in section 4).

Axiom 2 — There is a symmetric covariant derivative ∇ on S_5. (as in section 4).

Axiom 3 — The 1-form field dn and the vector field N should be covariantly constant (as in section 4):

$$\nabla dn = \nabla N = 0.$$

Remark : The axiom 3 implies that ∇ induces a symmetric covariant derivative $\nabla_{(n)}$ on $S_4(n)$ for each $n \in I$.

Definition — A vector u will be called 'natural' if

$$\langle\, dn, u \,\rangle = 0.$$

Remark : From axiom 3, it follows that if u is a natural vector field, and if p is any vector, then $\nabla_p u$ is also a natural vector. Also, it is easy to see that a natural vector remains natural when parallel transported along an event-curve.

Axiom 4 — On S_5, there is a 'metric' g for natural vectors only. g is compatible with ∇ in the sense that (i) if u and v are natural vector fields and p is any vector then

$$\nabla_p g(u, v) = g(\nabla_p u, v) + g(u, \nabla_p v) \qquad \ldots(55)$$

and (ii) $\nabla_{(n)}$ should be obtainable from the metric $g_{(n)}$ (that is induced by g on $S_4(n)$) by the usual formula for the Christoffel symbols in terms of the metric. We also require $g_{(n)}$ to be non-degenerate and of signature 2 for each n.

Axiom 5 —.Let C be a null geodesic lying in some $S_4(a)$ where $a \in I$. Then the projection of C, along event curves, on each space-time $S_4(n)$ should be a null geodesic in $S_4(n)$.

Axiom 5 is crucial to our argument. We justify it by noting that the ratio m_G/m_I is undefined for a photon, so a photon should not be confined to any particular space-time $S_4(a)$, but should be common to all the $S_4(n)$. So, the path of a photon in S_5 should not be regarded as a single curve, but a 2-dimensional 'ruled surface' with event curves as its 'generators'.

Remark : It follows from axiom 5 that if u is a natural vector that is null, then all the vectors obtained by parallel transporting u along its event curve are also null.

Now choose $a \in I$. From the above remark and Lemma 1 of the section, it follows that the induced metric $g_{(n)}$ on $S_4(n)$ for any $n \in I$ is related to the $g_{(a)}$ on $S_4(a)$ by

$$g_{(n)} = f\, g_{(a)} \qquad ...(56)$$

where $f: S_5 \to \mathbf{R}$ is a non-zero function, in the sense that if u, v are natural vectors at a point $\mathcal{P} \in S_4(n)$, and if u', v' are natural vectors on $S_4(a)$ that are obtained by parallel transporting u and v along the event curve passing through $\mathcal{P}$, then

$$g_{(n)}(u, v) = f(\mathcal{P})\, g_{(a)}(u', v'). \qquad ...(57)$$

Next we show that f is a function of n alone.

Definition : A coordinate system X^α on some neighbourhood in S_5 will be called 'nice' if $X^5 = n$ and

$$\frac{\partial}{\partial X^5} = N.$$

Remark : From axiom 3, it follows that in nice coordinates,

$$\Gamma^5_{\alpha\beta} = \Gamma^\alpha_{5\beta} = \Gamma^\alpha_{\beta 5} = 0 \;\forall\; \alpha, \beta. \qquad ...(58)$$

Now consider any 4 linearly independent natural null vectors $e_1,\ldots,e_4$ at a point $\mathcal{P}_a \in S_4(a)$ so $\{e_1,\ldots, e_4\}$ is a null basis for $S_4(a)$ at the point $\mathcal{P}_a$. Let X^α, $(\alpha \leqslant 4)$, be the corresponding Riemannian normal coordinates in some neighbourhood $\triangle$ of $\mathcal{P}_a$ in $S_4(a)$. In this coordinate system, for at $\alpha, \beta, \gamma \leqslant 4$,

$$g_{(a)\alpha,\beta,\gamma} = 0 \qquad ...(59)$$

and so the coefficients $\underset{(a)}{\Gamma}{}^\alpha_{\beta\gamma}$, $(\alpha, \beta, \gamma \leqslant 4)$, of $\nabla_{(a)}$ are zero as by axiom 4, for $\alpha, \beta, \gamma \leqslant 4$,

$$\underset{(a)}{\Gamma}{}^\alpha_{\beta\gamma} = \tfrac{1}{2}\, g^{\alpha\mu}_{(a)} \left(g_{(a)\mu\gamma,\beta} + g_{(a)\mu\beta,\gamma} - g_{(a)\beta\gamma,\mu}\right) \quad (\alpha, \beta, \gamma \leqslant 4). \qquad ...(60)$$

Also, as the basis $\{e_\alpha\}$ $\alpha \leqslant 4$ is null,

$$g_{(a)\alpha\alpha} = 0, \; (\alpha \leqslant 4). \qquad ...(61)$$

Now, project the coordinates X^α, $(\alpha \leqslant 4)$, along event curves passing through $\triangle$, on each $S_4(n)$, and take $X^5 = n$, so as to obtain a nice coordinate system X^α $(\alpha \leqslant 5)$ in the neighbourhood $\triangle \times I$ in S_5. Let $\mathcal{P}_n$ denote the projection of $\mathcal{P}_a$ on

$S_4(n)$. In $S_4(a)$, the coordinate curves corresponding to X^α, $(\alpha \leqslant 4)$, passing through $\mathcal{P}_a$, are null geodesics as X^α, $(\alpha \leqslant 4)$, are Riemann normal coordinates with $\mathcal{P}_a$ as the origin and e_α, $(\alpha \leqslant 4)$, are null vectors. So the same should be true in each $S_4(n)$ by axiom 5 and the construction of the nice coordinate system X^α $(\alpha \leqslant 5)$. Hence, on $S_4(n)$, $\underset{(n)}{\Gamma^1_{11}}$ should be zero at the point $\mathcal{P}_n$. Now, we know that

$$g_{(n)} = f\, g_{(a)}$$

and

$$g_{(a)\alpha\beta,\gamma} = 0 \quad (\alpha, \beta, \gamma \leqslant 4).$$

Hence,

$$\underset{(n)}{\Gamma^1_{11}} = \tfrac{1}{2}\, g^{1\mu}_{(n)} \, (2g_{(n)1\mu,1} - g_{(n)11,\mu}) = \frac{f_{,1}}{f}. \qquad \ldots(62)$$

Hence, $$\frac{\partial f}{\partial X^1} = 0. \qquad \ldots(63)$$

Similarly, $$f_{,2} = f_{,3} = f_{,4} = 0. \qquad \ldots(64)$$

Hence, f is a function of n alone, i.e., $f = f(n)$.

Hence, $$g_{(n)} = f(n)\, g_{(a)}. \qquad \ldots(65)$$

Hence, any geodesic in $S_4(a)$, when projected along event curves to $S_4(n)$ will be a geodesic in $S_4(n)$.

Now note that if two hypothetical particles A and B with different values a and b of m_G/m_I are released from rest in an experiment performed on board of a freely falling spaceship orbiting the earth, such that m_G/m_I for the spaceship is a, then the particle A will be 'unaccelerated' while the particle B will be 'accelerated' with respect to the spaceship. This contradicts our result that geodesics in $S_4(a)$ and $S_4(b)$ are obtainable from each other by projection along event curves. This shows that the 5-dim. generalization of general relativity (or similar 'metric' theories) is not possible.

6. Conclusions

We feel that equivalence principle (that m_G/m_I = constant) is foreign to the structure of Newtonian gravitation. Cartan, influenced by general relativity, tried to give it an important role by rewriting Newton's theory in a geometric form. Cartan's theory weakened the concept of an inertial frame: note that two Galilean frames could be accelerated with respect to each other in Cartan's theory. This would not have been acceptable to Newton.

In this note, we have (*i*) removed the equivalence principle from the list of axioms, (*ii*) but have maintained a geometric form compatible with general covariance, and *(iii)* in the process restored the inertial frames to their Newtonian status in the sense that they move with uniform velocities with respect to each other.

Of course, if the principle of equivalence is valid emperically, (or if we restrict our attention to the intrinsic geometry of the hypersurface $S_4(a)$ where $a \neq 0$) then the Newtonian inertial frames cannot be realized except in an 'island' universe (Misner *et al.* 1973). However, the interesting point is that the equivalence principle can be bypassed in the case of Cartan's theory by a 5-dimensional generalization.

On the other hand, it has been shown in section 5 that the equivalence principle is fundamental to general relativity — it cannot be bypassed by such a 5-dim. generalization. This, we feel, is an important difference between the theories of Cartan and Einstein.

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Reference

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). Gravitation. W. H. Freeman and Co., San Francisco, Chapter 12.