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A STUDY OF COST VARIATIONS IN EXTREME POINT MATHEMATICAL PROGRAMMING PROBLEMS*

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Equipped with an optimal solution to a given extreme point mathematical programming problem (E.P.M.P.P.) (Kirby *et al.* 1972a, b) we test here the sensitivity of this solution subject to variations in the cost (price) vector. A critical perturbation in the cost (price) vector beyond which optimality of the given solution is lost has been found. In support of the theory developed, a numerical example has also been added.

INTRODUCTION

In Kirby *et al.* (1972a) an extreme point mathematical programming problem (E.P.M.P.P.)

$$\begin{aligned} &\text{Max. } CX \\ &\text{subject to } AX = b \end{aligned} \qquad \text{...(P.1)}$$

and X is an extreme point of

$$\begin{aligned} DX &= d \\ X &\geqslant 0 \end{aligned}$$

has been solved by ranking extreme points of the linear programming problem

$$\begin{aligned} &\text{Max. } CX \\ &\text{subject to } DX = d \\ &\qquad X \geqslant 0 \end{aligned} \qquad \text{...(P.2)}$$

systematically starting from optimal extreme point, till feasibility in $AX = b$ is achieved.

In the above A is $m \times n$, b is $m \times 1$, D is $p \times n$, d is $p \times 1$, X and 0 are $n \times 1$ matrices.

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Many practical problems can be tackled successfully by techniques evolved for solving the extreme point mathematical programming problem (P.1) (Kirby and Scobey 1970). With a very common phenomenon of ever changing prices, a post optimal study of an E.P.M.P.P. is obviously an important study of problems we are frequently made to face. For this we test sensitivity of a linear programming problem (Barnett 1962, Courtillot 1962, Shetty 1961) with one additional condition that the resulting solution be achieved at an extreme point of a given convex polyhedron.

Let X^0 be an extreme point of (P.2) which gives an optimal solution of (P.1), and let $CX^0 = Z_0$. Let cost vector C be changed to $C^+ = C + \varphi f$, where φ is a non-negative scalar and f is an arbitrary but specified vector. Clearly changes in C for a specified f are controlled by values of φ.

It being wasteful to solve the perturbed problem from scratch, we give here the possibilities and means of getting from optimal solution of the original E.P.M.P.P., an optimal solution of the E.P.M.P.P. with changed prices. Critical value of φ (called φ_c), beyond which any change in c would render X^0 non-optimal has been found. Again, when $\varphi > \varphi_c$ i.e. when the original optimal solution X^0 is rendered non-optimal, a procedure for reaching an optimal solution of the E.P.M.P.P. with changed cost vector i.e. c^+, has been evolved.

Throughout this paper we have proceeded with an assumption of no-degeneracy, which is justifiable as almost all practical problems are non-degenerate.

PROCEDURE

Introduce a cut $CX \leqslant Z_0$ and consider the linear programming problem

$$\begin{aligned} &\text{Max. } CX \\ &\text{s.t. } DX = d \\ &\quad CX \leqslant Z_0 \\ &\quad X \geqslant 0 \end{aligned} \qquad \text{...(P.3)}$$

clearly (P.3) is a constrained form of (P.2). Let X^* be one of the optimas of (P.3) alternate to X^0.

Changing the cost vector to $C^+ = C + \varphi f$, (where $\varphi > 0$ is taken arbitrarily small) in (P.3), let us consider the linear programming problem :

$$\begin{aligned} &\text{Max. } C^+X \\ &\text{s.t. } DX = d \\ &\quad CX \leqslant Z_0 \\ &\quad X \geqslant 0 \end{aligned} \qquad \text{...(P.4)}$$

For $\varphi = 0$, (P.4) becomes (P.3) and so X^0 and X^* are both optimal extreme point solutions of (P.4). But for $\varphi \neq 0$, the hyperplane $Z = C^+X$ representing the objective function gets tilted and so one of these is bound to loose its optimality character. For $\varphi \neq 0$ and a specified f let when X^0 loses its optimality character, X^* be one of the alternatives which continue to be optimal for (P.4).

We have not yet given any definite value to φ. Now choose $\varphi > 0$, such that we have for (P.4)

$$(\alpha) \begin{cases} \text{(i)} & X^0 \text{ as a second best extreme point solution, and} \\ \text{(ii)} & \text{another second best extreme point solution } \bar{X} \text{ (say) which is feasible} \\ & \text{for (P.1).} \end{cases}$$

Such a choice of φ is made by method '*S*' given below.

Method 'S'

Let B be a basis for optimal extreme point solution X^* of (P.4). Write

$$H(B) = \{j \mid Z_j^+ - C_j^+ > 0\}$$

$$\theta_j = \operatorname*{Min}_i \left\{\frac{x_{B_i}}{y_{ij}} \,\middle|\, y_{ij} > 0\right\},\ j \in H(B)$$

$$\gamma_B = \operatorname*{Min}_{j \in H(B)} \{\theta_j(Z_j^+ - C_j^+) \mid \theta_j > 0\}$$

$$\delta = \operatorname{Min}\{\gamma_B \mid B \text{ is a basis for } X^*\}$$

where $Z_j^+ - C_j^+$ are values of $(Z_j - C_j)$ evaluated for cost vector C^+.

x_{B_i} : ith component of X_B where $X^* = [X_B, 0]$

y_{ij} : ith component of y_j where $y_j = B^{-1}d_j$

d_j : jth column of the coefficient matrix in (P.4).

If $\delta = \theta_k(Z_k^+ - C_k^+)$, then by inserting d_k into the basis we get a second best extreme point solution. If the above minima are not determined uniquely we are bound to get more than one second best extreme point solution of (P.4) (Hadley 1962). As in the present case we are ourselves interested in getting more than one second best extreme point solutions, we choose φ as follows :

When δ is determined uniquely — This is the case representing existence of a unique optimal basis of (P.4) yielding minimum γ_B. From simplex table corresponding to basis B for which such a δ is achieved we put for two different suffixes k and r in accordance with our requirement (α)

$$\gamma_B = \theta_k(Z_k^+ - C_k^+) = \theta_r(Z_r^+ - C_r^+) \qquad ...(1)$$

This equation determines the required value of φ in this case.

When δ is not determined uniquely — In this case there will be alternate optimal bases yielding same minimum γ_B. Let B_1 and B_2 be two bases for which

$$\gamma_{B_1} = \gamma_{B_2} = \delta.$$

Let $\gamma_{B_1} = \theta_k^1 (Z_k^{1^+} - C_k^{1^+})$

and $\gamma_{B_2} = \theta_r^2 (Z_r^{2^+} - C_r^{2^+})$

Then φ is determined by equating

$$\theta_k^1 (Z_k^{1^+} - C_k^{1^+}) = \theta_r^2 (Z_r^{2^+} - C_r^{2+}) \qquad ...(2)$$

(suffixes 1 and 2 correspond to bases B_1 and B_2 respectively).

X^0 necessarily is one of the second best extreme point solutions. Let us suppose that X^0 is obtained by inserting d_k into the basis for X^*. Also by inserting d_r we must get some other feasible point of (P.1).

We shall first proceed with the case when it is so (i.e. requirement α is satisfied completely) and then take up the case when the second best extreme point solution of (P.4) (other than X^0) comes out to be a point infeasible for (P.1).

Case I — Let by inserting d_r in an optimal basis (i.e. a basis for X^*) determined by δ we get $\bar{X}$ as a second best extreme point solution of (P.4) and let $\bar{X}$ be feasible for (P.1).

As $C^+ = C + \varphi f$

so $C_B^+ = C_B + \varphi f_B$

where f_B are components of f corresponding to basic variables then,

$$\begin{aligned} Z_j^+ - C_j^+ = C_B^+ y_j - C_j^+ &= (C_B + \varphi f_B) y_j - (C_j + \varphi f_j) \\ &= (C_B y_j - C_j) + \varphi(f_B y_j - f_j) \\ &= (Z_j - C_j) + \varphi(f_B y_j - f_j) \qquad ...(3) \end{aligned}$$

So from (1) we have

$$\theta_k [(Z_k - C_k) + \varphi(f_B y_k - f_k)] = \theta_r [(Z_r - C_r) + \varphi(f_B y_r - f_r)]$$

So,

$$\varphi = -\frac{\theta_k(Z_k - C_k) - \theta_r(Z_r - C_r)}{\theta_k(f_B y_k - f_k) - \theta_r(f_B y_r - f_r)} = \varphi_0 \qquad \ldots(4)$$

This is the critical value of φ in this case.

Again for the case when δ is not determined uniquely we have from (2) and (3)

$$\theta_k^1[(Z_k^1 - C_k^1) + \varphi(f_{B_1}y_k^1 - f_k)] = \theta_r^2[(Z_r^2 - C_r^2) + \varphi(f_{B_2}y_r^2 - f_r)]$$

then

$$\varphi = -\frac{\theta_k^1(Z_k^1 - C_k^1) - \theta_r^2(Z_r^2 - C_r^2)}{\theta_k^1(f_{B_1}y_k^1 - f_k) - \theta_r^2(f_{B_2}y_r^2 - f_r)} = \varphi_0 > 0 \qquad \ldots(5)$$

which is the critical value of φ when alternate optimal bases exist yielding same δ.

Case II — Let by inserting d_r into the basis we get $\bar{X}$ which is not feasible for (P.1).

In this case we make $\bar{X}$ play the role of X^*.

Introduce a cut $(C + \varphi_0 f) X \leqslant Z_1$ [where $(C + \varphi_0 f) X^0 = Z_1$, φ_0 being the value of φ given by (4) or (5) as the case may be] and consider the linear programming problem

$$\begin{aligned} &\text{Max } (C^+ + \varphi_1 f) X \\ &\text{s.t. } DX = d \\ &C^+X \leqslant Z_1 \\ &X \geqslant 0 \end{aligned} \qquad \ldots(\text{P.5})$$

where $C^+ = C + \varphi_0 f$ and φ_1 is a nonnegative scalar.

For $\varphi_1 = 0$, X^0 and $\bar{X}$ are optimal extreme point solutions of (P.5). But if $\varphi_1 \neq 0$, then X^0 ceases to be optimum and we are left with $\bar{X}$ as one of the optimal extreme point solutions of (P.5). Now we so choose φ_1 that we have for (P.5)

(i) X^0 as a second best extreme point solution, and

(ii) another second best extreme point solution X^1 (say) feasible for (P.1).

Choice of such a φ_1 is made again by method 'S'.

If by inserting d_k' (kth columns of the coefficient matrix in P.5) we get X^0 and by inserting d_r' we have X^1, then put

$$\theta_k(Z_k^{++} - C_k^{++}) = \theta_r(Z_r^{++} - C_r^{++}) \qquad \ldots(6)$$

$(Z_j^{++} - C_j^{++})$ is the value of $(Z_j - C_j)$ evaluated for C being taken as

$$C^{++} = C^{+} + \varphi_1 f = C + (\varphi_0 + \varphi_1) f.$$

Equation (6) gives

$$\varphi_1 = - \frac{\theta_k(Z_k^{+} - C_k^{+}) - \theta_r(Z_r^{+} - C_r^{+})}{\theta_k(f_B y_k - f_k) - \theta_r(f_B y_r - f_r)} = \varphi_1'$$

where f_B and θ_k, θ_r are evaluated corresponding to basis B for $\bar{X}$.

When δ for (P.5) is not determined uniquely we proceed as in Case I and obtain

$$\varphi_1 = - \frac{\theta_k^1 (Z_k^{1^+} - C_k^{1^+}) - \theta_r^2 (Z_r^{2^+} - C_r^{2^+})}{\theta_k^1 (f_{B_1} y_k^1 - f_k) - \theta_r^2 (f_{B_2} y_r^2 - f_r)} = \varphi_1' \cdot$$

Here B_1 and B_2 are two different bases yielding same δ. Also suffixes 1 and 2 indicate the various terms being evaluated for B_1 and B_2 respectively.

$\varphi^1 = \varphi_0 + \varphi_1'$ gives the critical values of φ in this case.

If X^1 again comes out to be infeasible for (P.1) we repeat the process and continue till an extreme point of $DX = d$, $X \geqslant 0$ feasible for (P.1) as required is obtained.

Whenever φ is taken less than or equal to its critical value φ_c [φ_0 in case (i), φ^1 in case (ii)], X^0 the given optimal solution of (P.1), continuous to be optimal for the perturbed problem. But the moment $\varphi > \varphi_c$, X^0 ceases to be optimal for the perturbed problem. With optimal solution as X^* [case (i)] or $\bar{X}$ [case (ii)]; that second best solution which is other than X^0 and is feasible for $AX = b$, gives the new optimal solution of the perturbed problem for φ slightly exceeding φ_c.

REMARKS

(1) Whenever f is normal to the optimal hyperplane at an extreme point of $DX = d$, $X \geqslant 0$ feasible for (P.1), then this extreme point solution remains optimal for every $\varphi > 0$.

(2) Whenever optimal solution for (P.1) is unique there exists an interval for φ viz. $0 \leqslant \varphi \leqslant \varphi_c$ for which X^0 remains optimal. When optimal solution of (P.1) is not unique, we will have $\varphi_0 = 0$.

(3) Sensitivity of the cost vector is tested for only rational changes in C. When the changes are drastic it is always better to solve the problem afresh.

ALGORITHM

The complete procedure can be incorporated in the following working steps:

Step 1 : Solve (P.3). X^0 is one of its optimal extreme point solution and $CX^0 = Z^0$. Find alternative optima and let X^* be one of the alternatives found.

Step 2 : Solve (P.4). At least one of the extreme point solutions of (P.3) alternate to X^0 will be its optimal solutions.

Step 3 : In the optimal simplex table obtained in step 2, apply method 'S' and determine φ_0 as given by (4) or (5) as the case may be.

Let by entering columns corresponding to kth and rth suffixes we obtain solutions X^0 and $\bar{X}$ respectively. Clearly $\bar{X}$ is an extreme point of $DX = d$, $X \geqslant 0$ which may or may not be feasible for $AX = b$. If $\bar{X}$ is feasible for (P.1), we have obtained the critical value of φ. If $\bar{X}$ is not feasible for (P.1) go to step 4.

Step 4 : Repeat steps 1–3 by considering (P.3)′ and (P.5) in place of (P.3) and (P.4) in steps 1 and 2 respectively i.e. consider

$$\begin{aligned} &\text{Max } C^+X \\ &\text{s.t. } DX = d \\ &\quad C^+X \leqslant Z_1 \\ &\quad\quad X \geqslant 0 \end{aligned} \qquad \text{...(P.3)}'$$

and

$$\begin{aligned} &\text{Max } (C^+ + \varphi_1 f)\, X \\ &\text{s.t. } DX = d \\ &\quad C^+X \leqslant Z_1 \\ &\quad\quad X \geqslant 0 \end{aligned} \qquad \text{...(P.5)}$$

where $C^+ = C + \varphi_0 f$ and $(C + \varphi_0 f)\, X^0 = Z_1$.

Continue repeating step 4 till such an $\bar{X}$ comes out to be feasible for (P.1).

For finding new optimal solution of the perturbed problem :

Second best extreme point solutions (other than X^0) of (P.4) or (P.5) as the case may be determine the new optimal solutions of (P.1) with changed cost vector.

EXAMPLE

Consider the extreme point mathematical programming problem

$$\begin{aligned} \text{Max. } & x_1 + 2x_2 \\ \text{subject to } & 2x_1 + 18x_2 \leqslant 221 \\ & 6x_1 + \ \ 2x_2 \leqslant \ \ 91 \end{aligned}$$

and (x_1, x_2) is an extreme point of

$$\begin{aligned} -4x_1 + x_2 &\leqslant 4 \\ -x_1 + x_2 &\leqslant 7 \\ -x_1 + 7x_2 &\leqslant 73 \\ 3x_1 + 2x_2 &\leqslant 57 \\ 2x_1 - x_2 &\leqslant 17 \\ 5x_1 - 7x_2 &\leqslant 20 \\ x_1, x_2 &\geqslant 0 \end{aligned} \qquad \text{...(P.1)}$$

Let us discuss sensitivity of the given optimal solution $X^0 = [x_1 = 4, x_2 = 11]$ of (P.1), given that $f = [1 \ -3]$.

After adding slack variables (P.3) takes the form :

$$\begin{aligned} &\text{Max. } x_1 + 2x_2 \\ \text{subject to } &-4x_1 + x_2 + x_3 = 4 \\ &-x_1 + x_2 + x_4 = 7 \\ &-x_1 + 7x_2 + x_5 = 73 \\ &3x_1 + 2x_2 + x_6 = 57 \\ &2x_1 - x_2 + x_7 = 17 \\ &5x_1 - 7x_2 + x_8 = 20 \\ &x_1 + 2x_2 + x_9 = 26 \\ &x_1, x_2, \dots, x_9 \geqslant 0 \end{aligned} \qquad \text{...(P.3)}$$

$X^* = [x_1 = 12, \ x_2 = 7]$, an optimal solution of (P.3) alternate to X^0 is given by Table I.

TABLE I

f_B	C_B	B	X_B	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
-3	2	d_2	7	0	1	0	0	0	0	$-1/5$	0	$2/5$
1	1	d_1	12	1	0	0	0	0	0	$2/5$	0	$1/5$
0	0	d_3	45	0	0	1	0	0	0	$9/5$	0	$2/5$
0	0	d_6	7	0	0	0	0	0	1	$-4/5$	0	$-7/5$
0	0	d_5	36	0	0	0	0	1	0	$9/5$	0	$-13/5$
0	0	d_8	9	0	0	0	0	0	0	$-17/5$	1	$9/5$
0	0	d_4	12	0	0	0	1	0	0	$3/5$	0	$-1/5$
		$Z_j - C_j =$		0	0	0	0	0	0	0	0	1
		$f_B y_j - f_j =$		0	0	0	0	0	0	1	0	-1
		$\theta_j =$								20		5

From (4), critical value of φ is given by

$$\varphi_c = -\frac{\theta_7(Z_7 - C_7) - \theta_9(Z_9 - C_9)}{\theta_7(f_B y_7 - f_7) - \theta_9(f_B y_9 - f_9)}$$

$$= -\frac{20(0) - 5(1)}{20(1) - 5(-1)} = \tfrac{1}{5}$$

If we enter d_7 into the basis for X^* and depart d_5, we get back the given original optimal solution X^0. If we enter d_9 into the basis for X^* and depart d_8, we get $\bar{X} = [x_1 = 11, x_2 = 5]$, which is feasible for (P.1).

Hence we conclude that for the perturbed problem

$$\text{Max. } (1 + \varphi)\, x_1 + (2 - 3\varphi)\, x_2$$

subject to constraints given in (P.1),

X^0 remains optimal for $\varphi \leqslant \frac{1}{5}$.

But the moment φ is taken slightly greater than $\frac{1}{5}$ $(= \varphi_c)$, $\bar{X}$ becomes the new optimal solution.

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