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ON THE COMPLEMENTARITY PROBLEM OVER POLYHEDRAL CONE

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Let S be a pointed, solid polyhedral cone in R^n, S^* its polar cone, and $g : S \to R^n$ a map. The complementarity problem over S is that of finding a solution to the system

$$x \in S, g(x) \in S^*, \langle g(x), x \rangle = 0.$$

In this paper, the authors study the extent to which the existence of an $\bar{x} \in S$ with $g(\bar{x}) \in S^*$ implies the existence of a solution to the above problem.

1. INTRODUCTION

Several problems arising in various fields, such as mathematical programming, game theory, economic equilibrium, mechanics, structural engineering, etc., can be stated in the following unified form : Given a map $g : R^n_+ \to R^n$, find an $x \in R^n$ such that

$$x \geqq 0, \ g(x) \geqq 0, \ \langle g(x), x \rangle = 0. \qquad ...(1)$$

This problem is known as the complementarity problem (CP). During the past decade, a number of important results have been established, dealing with both the question of existence of a solution and that of constructing a solution [see, for example, Kojima (1975), Megiddo and Kojima (1977) and the references therein] to the problem (1).

In this paper, we consider the following generalized complementarity problem (GCP) : Given a pointed, solid polyhedral cone S, in R^n, its polar S^* and a mapping $g : S \to R^n$, find an x such that

$$x \geqq^S 0, \ g(x) \geqq^{S^*} 0, \ \langle g(x), x \rangle = 0 \qquad ...(2)$$

where $\geqq^S$ and $\geqq^{S^*}$ are the partial orderings generated by S and S^*. We first show the existence of a solution to variational inequality for independent, solid polyhedral cone, and then use this result to establish an existence theorem for the GCP (2). An application of the above existence theorem to a convex minimization problem is given at the end.

2. NOTATIONS AND DEFINITIONS

Let $\langle x, y \rangle$ denote the usual inner product of $x, y \in R^n$. For any $x \in R^n$, $\| x \|_\infty = \max \{ | x_i | : 1 \leqq i \leqq n\}$ denotes the l^∞-norm of x, and for an $n \times k$ matrix B, $\| B \|_\infty = \sup \{ \| Bt \|_\infty : \| t \| \leqq 1\}$ denotes the l^∞-norm of B.

A nonempty subset S of R^n is a polyhedral cone if there is a positive integer k and an $n \times k$ matrix B such that $S = \{Bt : t \in R^k_+\}$. A cone S is said to be pointed if $S \cap (-S) = \{0\}$, and is said to be solid if it has a nonempty interior. A polyhedral cone $S = \{Bt : t \in R^k_+\}$ in R^n is said to be independent if the columns of the matrix B are linearly independent.

The polar of $S = \{Bt : t \in R^k_+\}$ in R^n is the cone S^* defined by

$$S^* = \{y \in R^n : \langle x, y \rangle \geqq 0 \text{ for all } x \in S\}$$

or equivalently by $S^* = \{y \in R^n : B^T y \geqq 0\}$. The interior of S^*, if it exists, is given by $\text{int}\, S^* = \{y \in R^n : \langle x, y \rangle > 0 \text{ for all } 0 \neq x \in S\}$ or equivalently by $\text{int}\, S^* = \{y \in R^n : B^T y > 0\}$ if S is independent. For a listing of some of the properties of polyhedral cones and their polars that will be pertinent here, we refer to Bazaraa and Shetty (1976).

The partial ordering induced by the pointed, solid cone S will be denoted by $\geqq^S$, $\geqslant^S$ and $>^S$. That is,

$$x \geqq^S y \text{ iff } x - y \in S,\ x >^S y \text{ iff } x - y \in \text{int}\, S,$$
$$x \geqslant^S y \text{ iff } x - y \in S \text{ and } x \neq y.$$

In particular, we write $x \geqq^S 0$ ($x >^S 0$) iff $x \in S$ ($x \in \text{int}\, S$). The partial ordering for S^* is defined similarly.

A function $g : R^n \to R^n$ is said to be monotone on $D \subset R^n$ if

$$\langle g(x) - g(y), x - y \rangle \geqq 0 \text{ for all } x, y \in D.$$

It is said to be pseudomonotone if, for every $x, y \in D$,

$$(x - y)^T g(y) \geqq 0 \text{ implies } (x - y)^T g(x) \geqq 0.$$

Let $f : R^n \to R^m$ be a continuously differentiable function on R^n, and let S and Q be polyhedral cones, respectively, in R^n and in R^m. Then f is said to be Q-concave on S if, for each $x, y \in S$,

$$J_f(y)(x - y) + f(y) - f(x) \in Q.$$

3. Preliminary Results

Lemma 1 — Let $S = \{Bt : t \in R^k_+\}$ be an independent, solid polyhedral cone in R^n. Let $t^0 = \alpha e$ with a scalar $\alpha > 0$ and $e^T = (1, 1, \ldots, 1)$. Then :

(i) $x = Bt$, $t > 0$ implies $x >^S 0$.

(ii) If $x = Bt \in S$, $p = Bt^0 \in S$, and $p - x \in \partial S$

(i.e., the boundary of S), then $\| t \|_\infty = \alpha$.

PROOF : (i) Since S is solid, S^* is pointed. It is also true that S^* is a polyhedral cone and $(S^*)^* = S^{**} = S$. To prove (i) it suffices to show that

$$\langle y, Bt \rangle = t^T(B^T y) > 0$$

for all $0 \neq y \in S^*$. If $B^T y^0 = 0$ for some $0 \neq y^0 \in S^*$, then $B^T(-y^0) = 0$, and hence $-y^0 \in S^*$, contradicting the fact that S^* is pointed. Therefore $B^T y \neq 0$ for $0 \neq y \in S^*$. The desired result is now obvious from the definition of S^* and the fact that $t > 0$.

(ii) Let $x = Bt \in S$, and $p = Bt^0 \in S$. If, in addition, $p - x \in \partial S$, then $B(t^0 - t) \in \partial S \subset S$, and consequently, $0 \leqq t^0 - t = \alpha e - t$. Now if $\alpha e - t > 0$, i.e., $\alpha - t_i > 0$ for $1 \leqq i \leqq k$, then it will follow from (i) above that $p - x >^S 0$. Hence we conclude that $\| t \|_\infty = \max_i t_i = \alpha$.

Lemma 2 — Let $y^0 \in R^n$ be given, and assume S is an independent, solid polyhedral cone in R^n. If there is an $x^0 \geqq^S 0$ and a $p >^S x^0$ such that

$$\langle y^0, x - x^0 \rangle \geqq 0 \text{ for all } 0 \leqq^S x \leqq^S p,$$

then $$\langle y^0, x - x^0 \rangle \geqq 0 \text{ for all } x \geqq^S 0.$$

PROOF : Let $x \geqq^S 0$. Since $p >^S x^0$, we can choose a $0 < \lambda < 1$ sufficiently small so that $p \geqq^S (x^0 + \lambda(x - x^0)) = \lambda x + (1 - \lambda) x^0$. Since S is convex,

$$\lambda x + (1 - \lambda) x^0 \geqq^S 0.$$

Then

$$0 \leqq \langle y^0, \lambda x + (1 - \lambda) x^0 - x^0 \rangle = \lambda \langle y^0, x - x^0 \rangle,$$

and consequently, $\langle y^0, x - x^0 \rangle \geqq 0$ for all $x \geqq^S 0$.

Lemma 3 — Let $S = \{Bt : t \in R^k_+\}$ be an independent, solid polyhedral cone in R^n, and let $p >^S 0$. Then the set

$$S_p = \{x \in R^n : 0 \leqq^S x \leqq^S p\}$$

is a nonempty, compact and convex in R^n.

PROOF : Since $p >^S 0$, $0 \in S_p$. So S_p is nonempty. That S_p is convex is obvious from the convexity of S.

Let $\{y^j\}$ be a sequence is S_p that converges to y. Now for each $y^j \in \{y^j\}$,

$$\langle p - y^j, z \rangle \geqq 0 \text{ for all } z \geqq^{S^*} 0,$$

and since the inner product is continuous,

$$0 \leqq \lim_j \langle p - y^j, z \rangle = \langle p - \lim_j y^j, z \rangle = \langle p - y, z \rangle.$$

for all $z \geqq^{S^*} 0$, from which it follows that $p - y \in S$. But $\{y^j\}$ is also a sequence in S which is closed, and therefore $y \in S$. This shows that $y \in S_p$, and hence S_p is closed.

To prove the boundedness of S_p, let $x \in S_p$. Now we have $x = Bt$, $p = Bt^0$ for some $t, t^0 \in R^k_+$ which are uniquely determined since S is independent. We also have $p - x = B(t^0 - t) \in S$, which implies $t^0 - t \geqq 0$. Since all norms in R^n are equivalent, we can take l^∞-norm to accomplish the proof. In l^∞-norm, $t^0 - t \geqq 0$ implies $\| t \|_\infty \leqq \| t^0 \|_\infty$, and consequently,

$$\| x \|_\infty = \| Bt \|_\infty \leqq \| B \|_\infty \| t \|_\infty \leqq \| B \|_\infty \| t^0 \|_\infty,$$

from which it follows that S_p is bounded. Thus we prove that S_p is a nonempty, compact and convex set in R^n.

The next lemma is on variational inequalities, and was first stated and proved by Hartman and Stampacchia (1966).

Lemma 4 — Let C be a nonempty, compact and convex subset of R^n, and let $g : R^n \to R^n$ be continuous on C. Then there exits $x^0 \in C$ such that

$$\langle g(x^0), x - x^0 \rangle \geqq 0 \text{ for all } x \in C.$$

The following theorem extends the result of Lemma 4 to an independent, solid polyhedral cone, and gives the connection between variational inequalities and the GCP (2).

Theorem 1 — Let $S = \{Bt : t \in R^k_+\}$ be an independent, solid cone in R^n, and let ∂S denote the boundary of S. Suppose $g : R^n \to R^n$ is a continuous mapping on S. If there is a $u \geqq^S 0$ and a $p >^S u$ such that $\langle g(x), x - u \rangle \geqq 0$ for all $x \geqq^S 0$ with $p - x \in \partial S$, then there is an x^0 in S satisfying

$$\langle g(x^0), x - x^0 \rangle \geqq 0 \text{ for all } x \geqq^S 0. \quad ...(3)$$

Moreover, if S^* denotes the polar of S, then x^0 is a solution to the GCP (2).

PROOF: We have $p >^S 0$ since $p - u >^S 0$ and $u \geqq^S 0$. Now consider the set

$$S_p = \{x \in R^n : 0 \leqq^S x \leqq^S p\}.$$

According to Lemma 3, S_p is a nonempty, compact and convex set in R^n. Hene, it follows from Lemma 4 that there exists an $x^0 \in S_p$ with

$$\langle g(x^0), x - x^0 \rangle \geqq 0 \text{ for all } x \in S_p. \quad ...(4)$$

If $p >^S x^0$, then Lemma 2 with $y^0 = g(x^0)$ yields (3). Assume, thus, that $p - x^0 \in \partial S$. Then $\langle g(x^0), x^0 - u \rangle \geqq 0$, which together with (4) gives

$$\langle g(x^0), x - u \rangle \geqq 0 \text{ for all } x \in S_p.$$

But $p >^S u$, and thus Lemma 2 implies that

$$\langle g(x^0), x - u \rangle \geqq 0 \quad \text{for all} \quad x \geqq^S 0.$$

It is clear that $u \in S_p$. Now, $x = u$ in (4) yields $\langle g(x^0), u - x^0 \rangle \geqq 0$. Then adding the last two inequalities, we obtain (3). This completes the proof of the first part of the theorem.

Now assume $y = x + x^0$, $x \geqq^S 0$. It is clear that $y \geqq^S 0$. Then it follows from (3) that

$$0 \leqq \langle g(x^0), y - x^0 \rangle = \langle g(x^0), x \rangle \quad \text{for all} \quad x \geqq^S 0.$$

So $g(x^0) \geqq^{S^*} 0$ and in particular, $\langle g(x^0), x^0 \rangle \geqq 0$. Since $0 \in S$, $\langle g(x^0), x^0 \rangle \leqq 0$ also, and hence $\langle g(x^0), x^0 \rangle = 0$. Thus x^0 is a solution to the GCP (2).

4. Main Theorems

Now we give the following existence theorem for the GCP, as given by (2).

Theorem 2 — Let $S = \{Bt : t \in R^k_+\}$ be an independent, solid polyhedral cone in R^n, and let $g : R^n \to R^n$ be a continuous monotone function on S. If there is an $\bar{x} \geqq^S 0$ with $g(\bar{x}) >^{S^*} 0$, then the GCP (2) has a solution.

Proof: Since g is monotone,

$$\langle g(x), x - \bar{x} \rangle \geqq \langle g(\bar{x}), x - \bar{x} \rangle \quad \text{for each} \quad x \geqq^S 0.$$

As $\bar{x} = B\bar{t}$ for some $\bar{t} \in R^k_+$, we can write

$$\langle g(\bar{x}), x - \bar{x} \rangle = (t - \bar{t})^T (B^T g(\bar{x})),$$

in which $B^T g(\bar{x}) > 0$ since $g(\bar{x}) >^{S^*} 0$. Now choose a positive real number $\alpha > \alpha_0$, where

$$\alpha_0 = k \, \| \bar{t} \|_\infty \max_i \, (B^T g(\bar{x}))_i / \min_i \, (B^T g(\bar{x}))_i.$$

It is obvious that $\alpha > \| \bar{t} \|_\infty$. If we set $p = B(\alpha e)$, where $e^T = (1, 1, \ldots, 1)$, then $\alpha e - \bar{t} > 0$, and by Lemma 1(i), $p >^S \bar{x}$. An application of Lemma 1(ii) shows that $\| t \|_\infty = \alpha$ for all $x = Bt$ in S with $p - x \in \partial S$. Therefore, for all $x \geqq^S 0$ with $p - x \in \partial S$, we have

$$\begin{aligned}(t - \bar{t})^T B^T g(\bar{x}) &\geqq \| t \|_\infty \min_i \, (B^T g(\bar{x}))_i - k \, \| \bar{t} \|_\infty \max_i \, (B^T g(\bar{x}))_i \\ &= \alpha \min_i \, (B^T g(\bar{x}))_i - k \, \| \bar{t} \|_\infty \max_i \, (B^T g(\bar{x}))_i \\ &> 0.\end{aligned}$$

Thus we find a $p >^S \bar{x}$ such that, for all $x \geqq^S 0$ with $p - x \in \partial S$,

$$\langle g(x), x - \bar{x} \rangle \geqq 0.$$

Theorem 1 now gives the desired result.

Corollary 1 — The same conclusion holds if, instead, g is pseudomonotone on S.

PROOF: It is shown in the proof of Theorem 2 that there exists a $p >^S \bar{x}$ such that, for all $x \geqq^S 0$ with $p - x \in \partial S$, $\langle g(\bar{x}), x - \bar{x} \rangle \geqq 0$. Since g is pseudomonotone on S, $\langle g(\bar{x}), x - \bar{x} \rangle \geqq 0$ implies $\langle g(x), x - \bar{x} \rangle \geqq 0$. The existence of a solution to the GCP (2) is then guaranteed by Theorem 1.

Now, we apply Theorem 2 to the existence of a solution under feasibility assumption to the following convex programming problem: (P) Minimize $f_0(x)$ subject to $x \geqq^K 0$, $f(x) \geqq^{L^*} 0$, where K and L are independent, solid polyhedral cones, respectively, in R^n and in R^m, $f: R^n \to R^m$ is a continuously differentiable mapping L^*-concave on K, an $f_0 : R^n \to R$ is a continuously differentiable function convex (in ordinary sense) on K.

Theorem 3 — Let f_0, f, K and L be as in the statement of problem (P). Suppose that there is an $\bar{x} \geqq^K 0$ with $f(\bar{x}) >^{L^*} 0$ and $\nabla f_0(\bar{x}) >^{K^*} 0$. Then there exists an optimal solution to (P).

PROOF: It is easy to show that the product set $S = K \times L$ is an independent, solid polyhedral cone in R^{n+m}, and its polar is $S^* = K^* \times L^*$. Define the function $g(x, v) : S \to R^{n+m}$ by

$$g(x, v) = \begin{vmatrix} \nabla f_0(x) - J_f^T(x)\, v \\ f(x) \end{vmatrix},$$

where $J_f(x)$ is the Jacobian matrix of f at x. Taking convexity of f_0 and f into consideration, a simple calculation yields that $g(x, v)$ is monotone over S. It is also seen that $g(\bar{x}, 0) >^{S^*} 0$ for $(\bar{x}, 0) \geqq^S 0$. Now applying Theorem 2, we get a point (x^0, v^0) satisfying

$$\left.\begin{aligned} &(x^0, v^0) \geqq^S 0,\ g(x^0, v^0) \geqq^{S^*} 0, \\ &\langle g(x^0, v^0), (x^0, v^0) \rangle = 0. \end{aligned}\right\} \qquad \ldots(5)$$

Replacing S and g by their values in terms of K, L, f and f_0, (5) simplifies to

$$x^0 \geqq^K 0,\ v^0 \geqq^L 0,\ f(x^0) \geqq^{L^*} 0,$$

$$\nabla f_0(x^0) - J_f^T(x^0)\, v^0 \geqq^{K^*} 0,\ \langle f(x^0), v^0 \rangle = 0,$$

$$\langle \nabla f_0(x^0) - J_f^T(x^0)\, v^0, x^0 \rangle = 0,$$

which is the Kuhn-Tucker sufficient optimality criteria for (x^0, v^0) to be an optimal solution of (P). This completes the proof.

The following corollary may be useful.

Corollary 2 — Let f_0 be a continuously differentiable real valued function convex over an independent, solid polyhedral cone K in R^n. Suppose that the set $\{x \in R^n : x \geqq^K 0, \nabla f_0(x) >^{K*} 0\}$ is nonempty. Then problem (P_0) : Minimize $f_0(x)$ subject to $x \geqq^K 0$ has an optimal solution.

PROOF : It immediately follows from Theorem 3 when f and the corresponding cone L are deleted.

Remark 1 : According to Theorem 6 of Bazaraa and Goode (1973, p. 8), problem (P_0) has an optimal solution if f_0 is strongly convex over K. Corollary 2 presents an alternate set of hypotheses under which the same conclusion holds.

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