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ON PARA-SASAKIAN MANIFOLDS

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In this paper certain results on almost paracontact manifolds and its curvature tensors have been obtained. Para-Einstein manifold, which may be considered as analogous to Einstein manifold have been defined and studied.

1. INTRODUCTION

Let M be an n-dimensional differentiable manifold in which there are given a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η satisfying

$$\left.\begin{array}{l}\phi^2 X = X - \eta(X)\,\xi;\ \phi(\xi) = 0;\ \eta(\xi) = 1,\\ \eta(\phi X) = 0;\ \text{rank}\ \phi = n - 1.\end{array}\right\} \quad ...(1.1)$$

Then (ϕ, ξ, η) is called an almost paracontact structure and M an almost paracontact manifold. Further, if there is a Riemannian metric g in M such that

$$g(X, \xi) = \eta(X);\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \quad ...(1.2)$$

then M is called an almost paracontact metric manifold (Sato 1977).

An almost paracontact manifold is said to be normal (Sharfuddin 1977) if

$$N(X, Y) \overset{def}{=} [\phi, \phi]\,(X, Y) - d\eta(X, Y)\,\xi = 0 \quad ...(1.3)$$

where $[\phi, \phi]\,(X, Y) = \phi^2\,[X, Y] + [\phi X, \phi Y] - \phi\,[\phi X, Y] - \phi\,[X, \phi Y]$.

The 2-form F defined as $F(X, Y) = g(\phi X, Y)$ is symmetric.

If, in an almost paracontact metric manifold

$$(D_X\phi)\,(Y) = \eta(Y)\,X - 2\eta(X)\,\eta(Y)\,\xi + g(X, Y)\,\xi, \quad ...(1.4)$$

then it is called a para-Sasakian manifold (Sharfuddin 1977, Sato 1977). In a para-Sasakian manifold (Sharfuddin and Husain 1978)

$$R(X, Y, \xi) = \eta(X)\,Y - \eta(Y)\,X \quad ...(1.5)$$

$$R(\xi, X, Y) = \eta(Y)\,X - g(X, Y)\,\xi \quad ...(1.6)$$

$$R_1(X, \xi) = (1 - n)\,\eta(X) \quad ...(1.7)$$

where R and R_1 are the Riemannian and Ricci curvature tensors respectively.

Projective curvature tensor W is given by

$$W(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1}\{R_1(Y, Z)X - R_1(X, Z)Y\} \quad ...(1.8)$$

Concircular curvature tensor V is given by

$$V(X, Y, Z) = R(X, Y, Z) - \frac{r}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\} \quad ...(1.9)$$

2. Some Results

Let M and $\bar{M}$ be two almost paracontact manifolds with the almost paracontact structures (ϕ, ξ, η) and $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ respectively. Consider the product manifold $M \times \bar{M}$ and let $M_m \times \bar{M}_{\bar{m}}$ be the tangent space of $M \times \bar{M}$ at $(m, \bar{m})$, $m \in M$ and $\bar{m} \in \bar{M}$. We define

$$J_{(m,\bar{m})}(X_m, \bar{X}_{\bar{m}}) = (\phi X_m - \bar{\eta}(\bar{X}_{\bar{m}})\, \xi_m, \bar{\phi}\bar{X}_{\bar{m}} - \eta(X_m)\, \bar{\xi}_{\bar{m}})$$

Then we can easily show that J is an almost product structure on $M \times \bar{M}$. Hence we have

Theorem 2.1 — Let M and $\bar{M}$ be almost paracontact manifolds. Then the product manifold $M \times \bar{M}$ has an almost product structure induced by the almost paracontact structures of M and $\bar{M}$.

Remark : The result is also true in case we define J as

$$J_{(m,\bar{m})}(X_m, \bar{X}_{\bar{m}}) = (\phi X_m + \bar{\eta}(\bar{X}_{\bar{m}})\, \xi_m, \bar{\phi}\bar{X}_{\bar{m}} + \eta(X_m)\, \bar{\xi}_{\bar{m}}).$$

Theorem 2.2 — In a normal almost paracontact manifold

$$(\mathcal{L}_\xi \eta)(X) = 0 \quad ...(2.1)$$

$$(\mathcal{L}_\xi \phi)(X) = 0 \quad ...(2.2)$$

Proof : Since the manifold is normal, putting $Y = \xi$ in (1.3), we get

$$-\phi[\phi X, \xi] + \phi^2[X, \xi] - d\eta(X, \xi)\,\xi = 0 \quad ...(2.3)$$

Applying η to the above equation we get (2.1). Further, (2.3) gives

$$\phi(\mathcal{L}_\xi \phi)(X) = 0.$$

Therefore

$$(\mathcal{L}_\xi \phi)(X) = \mu(X)\,\xi \quad ...(2.4)$$

for some function $\mu(X)$ and consequently

$$\eta((\mathcal{L}_\xi \phi)(X)) = \mu(X) \quad ...(2.5)$$

Now, $(\mathcal{L}_\xi(\eta \circ \phi))(X) = 0,$

that is

$$((\mathcal{L}_\xi\eta)\,\phi)(X) + \eta((\mathcal{L}_\xi\phi)(X)) = 0$$

which, in consequence of (2.1) and (2.5) give that $\mu(X) = 0$. Then, (2.4) shows that (2.2) hold which completes the proof.

3. Para-Sasakian Manifolds

Para-Sasakian manifolds correspond in certain sense to Sasakian manifolds on one hand and to almost product almost decomposable manifolds on the other. In this section we prove a result which gives the condition for a Riemannian manifold to be a para-Sasakian manifold.

Theorem 3.1 — Let M be a Riemannian manifold which admits a unit vector field ξ^i such that the 1-form $\eta_i = g_{ih}\xi^h$ satisfy

$$\eta \text{ is closed} \qquad \ldots(3.1)$$

$$\eta_{i,jk} = 2\eta_i\eta_j\eta_k - \eta_i g_{jk} - \eta_j g_{ik}. \qquad \ldots(3.2)$$

Then M admits a para-Sasakian structure.

Proof : We have $\xi^i\eta_i = 1$. Differentiating it covariantly, we get

$$\xi^i\eta_{i,j} = 0$$

and $$\xi^i\eta_{i,jk} = -\xi^i_{,k}\,\eta_{i,j}. \qquad \ldots(3.3)$$

Using (3.2), this reduces to

$$\xi^i_{,k}\,\eta_{i,j} = g_{jk} - \eta_j\eta_k. \qquad \ldots(3.4)$$

We put $\phi_{ij} = -\eta_{i,j}$. Then (3.4) reduces to

$$\xi^i_{,k}\,\phi_{ij} = g_{jk} - \eta_j\eta_k. \qquad \ldots(3.5)$$

Putting $\phi_{ij} = g_{ij}\phi^k_j$, we get

$$\phi^i_j\,\phi^j_k = \delta^i_k - \xi^i\eta_k. \qquad \ldots(3.6)$$

Therefore, ϕ^i_j, ξ^i, η_j and g_{ij} are structure tensors of an almost paracontact metric manifold. By using the fact that the 1-form is closed we get that ϕ_{ij} is symmetric. Hence M admits an almost paracontact metric structure (ϕ, ξ, η, g). Further,

$$\phi_{ij,k} = -\eta_{i,jk}$$

and therefore, eqn. (3.2) becomes

$$\phi_{ij,k} = \eta_i g_{jk} + \eta_j g_{ik} - 2\eta_i\eta_j\eta_k. \quad ...(3.7)$$

This completes the proof.

The following can easily be obtained.

Theorem 3.2 — In a para-Sasakian manifold

$$D_X\phi = -\phi X \quad ...(3.8)$$

$$N(X, Y) = 0. \quad ...(3.9)$$

4. Curvature Tensors

Theorem 4.1 — In a para-Sasakian manifold, we have

$$R(X, Y, \phi Z, W) - R(X, Y, Z, \phi W) = P(X, Y, Z, W) \quad ...(4.1)$$

where

$$P(X, Y, Z, W) = g(X, W) F(Y, Z) + g(X, Z) F(Y, W) - g(Y, W) F(X, Z) - g(Y, Z) F(X, W) + 2\{\eta(Y)\eta(Z) F(W, X) + \eta(W)\eta(Y) F(Z, X) - \eta(X)\eta(Z) F(W, Y) - \eta(W)\eta(X) F(Z, Y)\}. \quad ...(4.2)$$

Proof — From (1.4), we have

$$(D_X F)(Y, Z) = \eta(Y) g(\phi X, \phi Z) + \eta(Z) g(\phi X, \phi Y). \quad ...(4.3)$$

From the above equation, we get after some calculation

$$(D_X D_Y F)(Z, W) = \eta(Z)\{X . g(\phi Y, \phi W) - g(\phi Y, \phi D_X W)\} + \eta(W)\{X . g(\phi Y, \phi Z) - g(\phi Y . \phi D_X Z)\} - g(\phi Y, \phi W) F(X, Z) - g(\phi Y, \phi Z) F(X, W). \quad ...(4.4)$$

Using (4.3), (4.4) and the fact that

$$(D_X D_Y F)(Z, W) - (D_Y D_X F)(Z, W) - (D_{[X,Y]} F)(Z, W) = R(X, Y, \phi Z, W) - R(X, Y, Z, \phi W)$$

we get the required result.

Theorem 4.2 — Let M be a para-Sasakian manifold and let X, Y, Z, W be the tangent vectors of M orthogonal to ξ at every point m of M. Then

$$R(\phi X, \phi Y, \phi Z, \phi W) = R(X, Y, Z, W) \quad ...(4.5)$$

$$R(X, \phi X, Y, \phi Y) = R(X, Y, X, Y) - R(X, \phi Y, X, \phi Y) - 2\{g(X, Y) g(X, Y) - F(X, Y) F(X, Y)\}. \quad ...(4.6)$$

PROOF : We have from (4.1)

$$\begin{aligned} R(\phi X, \phi Y, \phi Z, \phi W) &= R(\phi X, \phi Y, Z, \phi W) + P(\phi X, \phi Y, Z, \phi W) \\ &= R(\phi X, \phi Y, Z, W) + P(\phi X, \phi Y, Z, \phi W) \\ &= R(Z, W, \phi X, \phi Y) + P(\phi X, \phi Y, Z, \phi W) \\ &= R(Z, W, X, \phi^2 Y) + P(Z, W, X, \phi Y) \\ &\quad + P(\phi X, \phi Y, Z, \phi W) \end{aligned}$$

which, in consequence of eqn. (4.2) and $\eta(X) = \eta(Y) = \eta(Z) = \eta(W) = 0$ gives

$$R(\phi X, \phi Y, \phi Z, \phi W) = R(Z, W, X, Y).$$

This proves (4.5). Further, we have from Bianchi identity

$$R(X, \phi X, Y, \phi Y) = - R(X, \phi Y, \phi X, Y) + R(X, Y, \phi X, \phi Y). \qquad \text{...(4.7)}$$

Using (4.5) in the above equation, we get

$$\begin{aligned} R(X, \phi X, Y, \phi Y) &= - R(X, \phi Y, X, \phi Y) + R(X, Y, X, Y) \\ &\quad - P(X, \phi Y, X, Y) + P(X, Y, X, \phi Y) \end{aligned}$$

which in consequence of (4.2) gives (4.6). Hence the proof is complete.

Nomizu (1968) has shown that the curvature transformation $R(X, Y)$ acts on the tensor algebra as a derivation and has discussed the condition $R(X, Y) . R = 0$ for hypersurfaces of Euclidean spaces, where as the condition $R(X, Y) . R_1 = 0$ has been discussed by Tanno (1969) for hypersurfaces of Euclidean spaces. In Sasakian manifolds these conditions were discussed by Takahashi (1969) and Tanno (1967), respectively. For a para-Sasakian manifolds, with these conditions, we prove the following results.

Theorem 4.3 — If a para-Sasakian manifold satisfies the condition

$$(R(X, \xi) . R) (Y, Z, W) = 0, \qquad \text{...(4.8)}$$

then the manifold is of constant curvature.

PROOF : Condition (4.8) can be written as

$$\begin{aligned} &R(X, \xi, R(Y, Z, W)) - R(R(X, \xi, Y), Z, W) \\ &\quad - R(Y, R(X, \xi, Z), W) - R(Y, Z, R(X, \xi, W)) = 0. \end{aligned}$$

Putting $Y = \xi$ in the above equation and using (1.5) and (1.6), we obtain after some calculation

$$R(X, Z, W) = g(X, W) Z - g(Z, W) X$$

which proves the statement.

Theorem 4.4 — If a para-Sasakian manifold M of dimension n satisfies the condition

$$(R(X, \xi) . R_1)(Y, Z) = 0, \quad \ldots(4.9)$$

then M is an Einstein manifold.

PROOF : Condition (4.9) can be written as

$$R_1(R(X, \xi, Y), Z) + R_1(Y, R(X, \xi, Z)) = 0.$$

Putting $Z = \xi$ and using (1.5), (1.6) and (1.7), we get after some calculation

$$R_1(X, Y) = -(n-1)\, g(X, Y).$$

Hence, the Scalar curvature $r = R_1(X_i, X_i) = -n(n-1)$ which gives that

$$R_1(X, Y) = \frac{r}{n}\, g(X, Y),$$

whereby proving the result.

We now state the following results whose proofs are simple.

Theorem 4.5 — If a para-Sasakian manifold is an Einstein one, the scalar curvature r has a negative constant value $-n(n-1)$.

Theorem 4.6 — If a para-Sasakian manifold is of constant sectional curvature, then it is of constant curvature -1.

Theorem 4.7 — A projectively flat para-Sasakian manifold is of constant curvature -1.

In what follows we prove the following :

Theorem 4.8 — If a para-Sasakian manifold M is concircularly flat, then the scalar curvature r is equal to $-n(n-1)$.

PROOF : As M is concircularly flat, $V(X, Y, Z)$ vanishes identically and we have

$$R(\xi, Y, Z, \xi) - \frac{r}{n(n-1)} \{g(Y, Z) - \eta(Y)\,\eta(Z)\} = 0$$

which, in consequence of (1.6) gives

$$\left(1 + \frac{r}{n(n-1)}\right) g(\phi Y, \phi Z) = 0.$$

Now, $g(\phi Y, \phi Z) = 0$ is not possible in general since g is non-singular and this proves the statement.

The following is direct.

Corollary 4.9 — A concircularly flat para-Sasakian manifold is necessarily a manifold of constant curvature -1.

We now prove

Theorem 4.10 — If in a para-Sasakian maifold

$$R_{ij} = bg_{ij} + c\phi_{ij}, \quad \ldots(4.10)$$

then

$$b = -(n-1) \quad \ldots(4.11)$$

$$c = \frac{1}{2(n-1)} c_{,k}\xi^k f \quad \ldots(4.12)$$

where R_{ij} is the Ricci curvature tensor and f is the trace of ϕ.

PROOF : Contracting (4.10) with ξ^i, we get $R_{ij}\xi^i = b\eta_j$ which, using (1.7) gives (4.11). Further, contraction of (4.10) gives

$$r = nb + cf.$$

Covariant differentiation of the above equation gives

$$r_{,k} = c_{,k}f. \quad \ldots(4.13)$$

We also have

$$r_{,k} = 2R^i_{k,i}. \quad \ldots(4.14)$$

Then (4.10), (4.13) and (4.14) give

$$c_{,k}f = 2(bg^i_k + c\phi^i_k)_{,i}$$

$$= 2c_{,i}\phi^i_k + 2c\phi^i_{k,i}$$

which in view of (1.4) becomes

$$c_{,k}f = 2c_{,i}\phi^i_k + 2c(n-1)\,\eta_k. \quad \ldots(4.15)$$

Contracting (4.15) with ξ^k, we get

$$c_{,k}\xi^k f = 2c(n-1)$$

which gives (4.12) whereby proving the theorem.

Definition — A para-Sasakian manifold M satisfying

$$R_1(X, Y) = bg(X, Y) + cF(X, Y) \quad \ldots(4.16)$$

where b is a constant and c a function on M, is called a para-Einstein manifold.

We now prove a result which gives a necessary and sufficient condition for a para-Sasakian manifold to be a para-Einstein manifold.

Theorem 4.11 — A para-Sasakian manifold M is para-Einstein manifold if and only if

$$(R(X, \xi) . R_1)(Y, Z) = c\,[\eta(Y) F(X, Z) + \eta(Z) F(X, Y)] \qquad ...(4.17)$$

holds for some function c on M.

PROOF : We have

$$(R(X, \xi) . R_1)(Y, Z) = -R_1(R(X, \xi) Y, Z) - R_1(Y, R(X, \xi) Z).$$

Now, let M be para-Einstein. Then using (4.16), we get

$$(R(X, \xi) . R_1)(Y, Z) = -c\,[R(X, \xi, Y, \phi Z) + R(X, \xi Z, \phi Y)].$$

Using (1.5) in the above equation, we have

$$(R(X, \xi) . R_1)(Y, Z) = -c\,[-\eta(Y) F(X, Z) - \eta(Z) F(X, Y)]$$

which gives (4.17).

Conversely, let (4.17) holds. Then putting $Z = \xi$ in (4.18), we have from Theorem 4.4

$$R_1(X, Y) + (n - 1)\, g(X, Y) = cF(X, Y)$$

which shows that M is a para-Einstein manifold. Hence the theorem is proved.

Remarks

(1) The name "Para-Einstein" manifold given to a para-Sasakian manifold satisfying the condition

$$R_1(X, Y) = bg(X, Y) + cF(X, Y)$$

is justified by the following.

We know from Theorem 4.4 that a para-Sasakian manifold satisfying the condition

$$(R(X, \xi) . R_1)(Y, Z) = 0$$

is an Einstein manifold. This condition read with Theorem 4.11 gives $c = 0$, and thus the para-Einstein manifold reduces to an Einstein manifold. This gives the analogy between a para-Einstein and an Einstein manifold.

(2) An alternative way of defining a manifold analogous to Einstein manifold in this setting could be by imposing the condition

$$R_1(X, Y) = bg(X, Y) + c\eta(X)\,\eta(Y)$$

on a para-Sasakian manifold. Such a structure has been studied by us in a different context in a paper to appear shortly.

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