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THE EVALUATION OF A LIMIT OF AN INDETERMINATE FORM

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It is of general interest to use a heuristic and classical approach to the evaluation of the pointwise limit

$$\lim_{\mathrm{Re}(\nu)\to 0^+}\left[\left(\frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{\nu-1} f(t)\,dt - f(x)\right)\Big/\nu\right] \qquad ...(1)$$

where f is continuous and differentiable on $(c.\ x)$, $-\infty < c < x$ and $\mathrm{Re}\,(\nu) > 0$.

The integral above is the Riemann-Liouville integral which defines integration and differentiation of arbitrary order and plays the central role in the fractional calculus. When $c = 0$ and $c = -\infty$ we have respectively Riemann's and Liouville's definitions for differintegration (Oldham Keith and Spanier 1974). One of the useful and popular operator notations for this integral is ${}_cD_x^{-\nu} f(x)$.

The limit of the difference quotient in (1) is described by Hille and Phillips (1957) as an 'infinitesmal generator'. These authors evaluated this limit in an L_p setting. The purpose of this paper is to present a method accessible to a wide class of readers with some familiarity with the gamma function.

The quotient in (1) is indeterminate when $\nu = 0$ because when $\nu = 0$ the above integral equals $f(x)$ (Ross 1975). The method used to resolve this indeterminacy is first to integrate by parts and then apply L'Hospital's rule. Integration by parts yields

$$\frac{f(c)\,(x-c)^\nu}{\Gamma(\nu+1)} + \frac{1}{\Gamma(\nu+1)}\int_c^x (x-t)^\nu f'(t)\,dt. \qquad ...(2)$$

Then (1) can be written as

$$\lim_{\mathrm{Re}\,(\nu)\to 0^+} \frac{f(c)\,(x-c)^\nu + \int_c^x (x-t)^\nu f'(t)\,dt - f(x)\,\Gamma(\nu+1)}{\nu\Gamma(\nu+1)}. \qquad ...(3)$$

When $\nu = 0$ we have the indeterminate form $[f(c) + f(x) - f(c) - f(x)]/0$ because $\Gamma(1) = 1$. L'Hospital's rule may be applied:

$$\lim_{\mathrm{Re}(\nu)\to 0^+} \frac{f(c)(x-c)^\nu \ln(x-c) + \int_c^x (x-t)^\nu \ln(x-t) f'(t)\,dt - f(x)\Gamma'(\nu+1)}{\Gamma(\nu+1) + \nu\Gamma'(\nu+1)}. \quad ...(4)$$

Passing to the limit gives the result

$$f(c) \ln (x-c) + \int_c^x \ln (x-t) f'(t)\, dt - f(x)\, \Gamma'(1) \quad ...(5)$$

where $\Gamma'(1)$ is the negative of Euler's constant.

Hille-Phillips (1975) in an L_p setting let the lower terminal of integration c in the R-L integral be equal to 0. The result is the same:

$$f(0) \ln x + \int_0^x \ln (x-t) f'(t)\, dt - f(x)\, \Gamma'(1). \quad ...(6)$$

In the L_p setting the derivative of the integral in (1) exists for all f in L_p for which $\int_0^x \ln (\nu - t) f(t)\, dt$ is in turn an integral of a function in L_p.

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