Indian J. pure appl. Math., **11**(7) : 856–862, July 1980

THE DI-BESSEL FUNCTION

HAROLD EXTON

27, Hollinhurst Avenue, Penwortham, Preston, Lancashire PR1 0AE, United Kingdom

(*Received* 1 *February* 1980)

A straightforward generalization of the Bessel function is proposed and an associated differential equation of the fourth order is given. It is shown that this differential equation is self-adjoint and that its solutions are orthogonal in a manner analogous to the Bessel functions. The possibility of expanding an arbitrary function in series of Di-Bessel functions is discussed. Two infinite integrals are evaluated. In conclusion, short tables of zeros and turning values are given and it is pointed out that any generalized hypergeometric function ${}_0F_3$ possesses an orthogonality property similar to that mentioned above.

1. INTRODUCTION

The Bessel function, whose manifold properties are well-known, possesses the series representation

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\nu+1+r)} (x/2)^{\nu+2r}. \qquad \ldots(1.1)$$

This function may be generalized giving

$$A_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r!\}^2\,\{\Gamma(\nu+1+r)\}^2} (x/2)^{\nu+2r} \qquad \ldots(1.2)$$

which we denote by the term Di-Bessel function.

If ν is an integer, we have the following generating function of the Di-Bessel function of integer order, conveniently referred to as the Di-Bessel coefficient:

$$I_0(2x^{1/2}t^{1/2})\,J_0(-2x^{1/2}t^{-1/2}) = \sum_{r=0}^{\infty} \frac{1}{\{r!\}^2} (x/2)^r\, t^r$$

$$\times \sum_{j=0}^{\infty} \frac{(-1)^j}{\{j!\}^2} (x/2)^j\, t^{-j}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{\{j!\}^2} (x/2)^j \sum_{n=-j}^{\infty} \frac{1}{\{(n+j)!\}^2} (x/2)^{n+j}\, t^n$$

$$= \sum_{n=-\infty}^{\infty} t^n \sum_{j=0}^{\infty} \frac{(-1)^j}{\{(n+j)!\}^2 \{j!\}^2} (x/2)^{n+2j}$$

$$= \sum_{n=-\infty}^{\infty} t^n A_n(x). \qquad ...(1.3)$$

This result corresponds to the well-known generating function for the Bessel coefficient

$$\exp(\tfrac{1}{2}zt - \tfrac{1}{2}zt^{-1}) = \sum_{n=-\infty}^{\infty} t^n J_n(x). \qquad ...(1.4)$$

Multiply both sides of (1.3) by t^{-n-1}, when a contour integral representation of the Di-Bessel coefficient is obtained by integrating around a contour which encircles the origin once in the positive direction. This is

$$A_n(x) = 1/(2\pi i) \int^{(0+)} t^{-n-1} I_0(2x^{1/2}t^{1/2})\, J_0(-2x^{1/2}t^{-1/2})\, dt. \qquad ...(1.5)$$

In this paper, it is taken that all quantities are real unless otherwise stated.

2. An Associated Differential Equation

From the series representation of $A(x)$, it is evident that

$$A_\nu(x) = (x/2)^\nu \{\Gamma(\nu+1)\}^{-2}\, {}_0F_3(-;\nu+1,\nu+1,1;-x^2/4). \qquad ...(2.1)$$

In investigating a differential equation associated with this function, we begin with the operational equation

$$\{\delta(\delta+b_1-1)(\delta+b_2-1)(\delta+b_3-1)-x\}\, y = 0. \qquad ...(2.2)$$

where $\delta \equiv xd/dx$.

If this equation is expanded, it takes the form

$$x^3y^{iv} + (b_1+b_2+b_3+3)\, x^2y''' + (b_1b_2+b_1b_3 + b_2b_3+b_1+b_2+b_3+1)\, xy'' + b_1b_2b_3y' - y = 0 \qquad ...(2.3)$$

a solution of which is

$${}_0F_3(-;b_1,b_2,b_3;x) \qquad ...(2.4)$$

[see Slater (1966, p. 42)].

If we put $b_1 = b_2 = \nu + 1$ and $b_3 = 1$, it follows that $(-2x)^{-1/2\nu} A_\nu(2ix^{1/2})$ is a solution of the equation

$$x^3 y^{iv} + 2(\nu + 3)\, x^2 y''' + \{(\nu + 2)(\nu + 3) + \nu + 1\}\, xy'' + (\nu + 1)^2\, y' - y = 0. \qquad \ldots(2.5)$$

Replace x by λx, and (2.5) takes the form

$$x^3 y^{iv} + 2(\nu + 3)\, x^2 y''' + \{(\nu + 2)(\nu + 3) + \nu + 1\}\, xy'' + (\nu + 1)^2\, y' - \lambda y = 0. \qquad \ldots(2.6)$$

3. The Orthogonality of the Di-Bessel Function

After a little manipulation, eqn. (2.6) may be expressed as

$$(x^{\nu+3} y'')'' + (\nu + 1)(x^{\nu+1} y')' - \lambda x^{\nu} y = 0 \qquad \ldots(3.1)$$

which is self-adjoint. By means of Green's formula, the solutions of (3.1) are orthogonal with respect to the weight function x^{ν} for $\nu > 0$. That is

$$\int_0^c x^{\nu} y(\lambda_i x)\, y(\lambda_j x)\, dx = 0, \; i \neq j \qquad \ldots(3.2)$$

where $\lambda_1, \lambda_2, \ldots$ are the solutions of

$$y(\lambda c) = 0 \qquad \ldots(3.3)$$

[see Ince (1926, p. 237)].

In order to evaluate the integral (3.2) when $i = j$, we write (3.1) in the form

$$p_0 y^{iv} + p_1 y''' + p_2 y'' + p_3 y' + p_4 y = 0 \qquad \ldots(3.4)$$

so that $p_0 = x^{\nu+3}$, $p_1 = 2(\nu + 3)\, x^{\nu+2}$, $p_2 = \{(\nu + 2)(\nu + 3) + \nu + 1\}\, x^{+1}$, $p_3 = (\nu + 1)^2\, x^{\nu}$ and $p_4 = -\lambda x^{\nu}$.

The integral (3.2) is equal to the bilinear concomitant of (3.1) taken between the limits 0 and c after division by $(\lambda_j - \lambda_i)$. This bilinear concomitant is

$$\begin{aligned} P(y_i, y_j) = {} & y_i \{p_3 y_j - d/dx(p_2 y_j) + d^2/dx^2(p_1 y_j) - d^3/dx^3(p_0 y_j)\} \\ & + dy_i/dx\, \{p_1 y_j - d/dx(p_1 y_j) + d^2/dx^2(p_0 y_j)\} \\ & + d^2 y_i/dx^2\, \{p_i y_j - d/dx(p_0 y_j)\} + d^3 y_i/dx^3\, \{p_0 y_j\}. \end{aligned} \qquad \ldots(3.5)$$

After some algebra, we have

$$\begin{aligned} P(y_i, y_j) = {} & (\nu + 1)(\nu + 2)(\nu + 3)\, x^{\nu} y_i y_j \\ & + \lambda_j \{2(\nu + 2)(\nu + 3) - (\nu + 1)\}\, x^{\nu+1} y_i y_j' \\ & + \lambda_j^2 (\nu + 2)\, x^{\nu+2} y_i y_j'' + \lambda_i (\nu + 1)\, x^{\nu+1} y_i' y_j \\ & + \lambda_i \lambda_j^2\, x^{\nu+3} y_i' y_j'' + \lambda_i^2 (\nu + 3)\, x^{\nu+2} y_i'' y_j \\ & - \lambda_j^2 \lambda_j x^{\nu+3} y_i'' y_j' + \lambda_i^3\, x^{\nu+3} y_i''' y_j' \end{aligned} \qquad \ldots(3.6)$$

primes denoting differentiations with respect to either $\lambda_i x$ or $\lambda_j x$ as appropriate.

The integral (3.2) when $i = j$ may now be calculated by taking the de l'Hôpital limit of

$$\frac{P(y_i, y_j)}{\lambda_j - \lambda_i} \qquad \ldots(3.7)$$

as $\lambda_j \to \lambda_i$, making use of the fact that

$$y(\lambda_i c) = 0 \qquad \ldots(3.8)$$

after letting $x \to c$. Hence,

$$- \int_0^c x^\nu \{y(\lambda_i x)\}^2 \, dx$$

$$= \lambda_i^2 \{2(\nu + 2)(\nu + 3) - (\nu + 1)\} c^{\nu+1} \{y_1'(c)\}^2 + \lambda_i^2 (\nu + 2) c^{\nu+2} y_i'(c) y_i''(c) + \lambda_i^2 c^{\nu+3} [\lambda_i^2 \{y_i''(c)\}^2 - y_i''(c) y_i'(c) + {}_i y_i'''(c) y_i'(c)]. \qquad \ldots(3.9)$$

Since $\quad y_i = {}_0F_3(-; \nu + 1, \nu + 1, 1; \lambda_i x) \qquad \ldots(3.10)$

we may calculate (3.9) explicitly, noting that

$$d^r/dx^r \, {}_0F_3(-; \nu + 1, \nu + 1, 1; x) = \frac{1}{\{(\nu + 1)_r\}^2 \, r!} \times {}_0F_3(-; \nu + r + 1, \nu + r + 1, r + 1; x). \qquad \ldots(3.11)$$

If $y(\lambda_i x)$ is replaced by its equivalent expression in terms of the Di-Bessel function, we have

$$\int_0^c x^\nu \{y(\lambda_i x)\}^2 \, dx = 2^{-\nu} \int_0^d x \{A_\nu(\mu_i x)\}^2 \, dx \qquad \ldots(3.12)$$

where $d = c^2$ and $\mu_i = \frac{1}{4}\lambda_i^2$, such that $A_\nu(\mu_i d) = 0$, noting that all the quantities $\{\lambda_i\}$ are negative.

Suppose that the function $f(x)$ possesses the Maclaurin expansion

$$f(x) = \sum_{r=0}^{\infty} b_r x^r \qquad \ldots(3.13)$$

but is otherwise arbitrary, then the expansion

$$f(x) = \sum_{r=0}^{\infty} c_r A_\nu(\mu_r x), \ \nu > 0 \qquad \ldots(3.14)$$

exists, where

$$c_r = \frac{\int_0^d x f(x)\, A_\nu(\mu_r x)\, dx}{\int_0^d x\, \{A_\nu(\mu_r x)\}^2\, dx} \quad \text{...(3.15)}$$

which is strictly analogous to the Fourier-Bessel expansion of $f(x)$. It is conjectured that $A_\nu(x)$ has an infinite number of real zeros and that each zero is simple with the possible exception of the origin.

4. Infinite Integrals

Consider, first of all, the integral

$$I_1 = \int_0^\infty e^{-ax} x^{\mu-1} A_\nu(bx)\, dx \quad \mathrm{Re}\,(a) > 0,\ \mathrm{Re}\,(\mu + \nu) > 0 \quad \text{...(4.1)}$$

which generalizes Hankel's integral

$$\int_0^\infty e^{-ax} x^{\mu-1} J_\nu(bx)\, dx. \quad \text{...(4.2)}$$

Replace $A_\nu(bx)$ by its series representation and integrate term-by-term, a process which is justified because the series in question converges uniformly for all finite values of its variable:

$$I_1 = \sum_{r=0}^{\infty} \frac{(-1)^r}{\{\Gamma(\nu+1+r)\}^2 \{r!\}^2} (b/2)^{\nu+2r} \int_0^\infty e^{-ax} x^{\mu+\nu+2r-1}\, dx$$

$$= a^{-\mu}(b/2a)^\nu \sum_{r=0}^{\infty} \frac{(-1)^r\, \Gamma(\mu+\nu+2r)}{\{\Gamma(\nu+1+r)\}^2 \{r!\}^2} (b/2a)^{2r}$$

$$= a^{-\mu}(b/2a)^\nu \frac{\Gamma(\mu+\nu)}{\{\Gamma(\nu+1)\}^2}\, {}_2F_3(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu, \tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2}; \nu+1, \nu+1, 1; -b^2/a^2). \quad \text{...(4.3)}$$

By suitably specializing the parameters, we have

$$\int_0^\infty e^{-ax} x^{1/2} A_{1/2}(bx)\, dx = (a\pi)^{-1} (2/b)^{1/2} \sin(2b/a) \quad \text{...(4.4)}$$

and

$$\int_0^\infty e^{-ax} x^{1/2} A_{-1/2}(bx)\, dx = (a\pi)^{-1} (2/b)^{1/2} \cos(2b/a). \quad \text{...(4.5)}$$

Similarly, it may be shown that

$$I_2 = \int_0^\infty e^{-ax} x^{\mu-1} A_\nu(b^2x^2/2)\, dx$$

$$= (b/2)^{2\mu} a^{-\mu-2\nu} \frac{\Gamma(\mu+2\nu)}{\{\Gamma(\nu+1)\}^2}$$

$$\times {}_4F_3\left(\begin{matrix}\tfrac{1}{4}\mu+\tfrac{1}{2}\nu, & \tfrac{1}{4}\mu+\tfrac{1}{2}\nu+\tfrac{1}{4}, & \tfrac{1}{4}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}, & \tfrac{1}{4}\mu+\tfrac{1}{2}\nu+\tfrac{3}{4}; \\ \nu+1, & \nu+1 & , & 1 & ;\end{matrix} - b^4/a^4\right) \quad \ldots(4.6)$$

provided by Re $(a) > 0$, Re $(\mu + 2\nu) > 0$ and $|b/a| \leqslant 1$.

5. Conclusion

Short tables of zeros, turning points and turning values

ν	zeros	turning values	turning points	ν	zeros	turning values	turning points
0	2.0344	−3.4975	3.5235	1.0	2.9225	0.7610	1.9116
	4.2055	37.2776	5.7850		5.2056	−3.6173	4.4834
	6.4123	−519.3956	8.0212		7.4491	32.8938	6.7900
	8.6269	8234.3580	10.2502		9.6820	−388.3032	9.0512
0.5	2.5131	0.7484	1.4211	1.5	3.2866	0.9007	2.3229
	4.7277	−3.0974	4.0235		5.6500	−5.0002	4.9129
	6.9467	29.7001	6.3024		7.9254	44.5456	7.2535
	9.1669	−374.8235	8.5480		10.1758	−502.9967	9.5342

ν	zeros	turning values	turning points
5.0	5.2068	0.1822	4.4280
	8.2020	−3.6436	7.4182
	10.7797	42.0013	10.0484
	13.2143	−473.7203	12.5182

The fact that the self-adjoint equation

$$(x^{\nu+3}y'')'' + (x^{\nu+1}y')' + x^\mu y = 0 \quad \ldots(5.1)$$

associated with the function ${}_0F_3$ with general values of its parameters shows that all these functions are orthogonal in a similar fashion to the solutions of (3.1), with the eigenvalues multiplicatively associated with the independent variable. Thus, further types of generalized Fourier-Bessel series involving hypergeometric functions of the fourth order, ${}_0F_3(-; b_1, b_2, b_3; x)$, exist.

In the case of the corresponding self-adjoint equation of order $2n$,

$$(x^{\nu+2n-1}y^{(n)})^{(n)} + \alpha_1(x^{\nu+2n-3}y^{(n-1)})^{(n-1)} + \ldots + \alpha_{n-1}(x^{\nu+1}y')' + x^{\mu}y = 0 \quad \ldots(5.2)$$

associated with the function ${}_0F_{2n-1}(b_1, b_2, \ldots, b_{2n-1}; x)$, the number of free parameters $b_1, b_2, \ldots$ is seen to be $(n + 1)$ only for such functions to afford a basis of series of generalized Fourier-Bessel type. It must be stressed that none of the parameters $\{b_i\}$ must be zero or a negative integer for the series ${}_0F_B$ to be defined. In all cases, however, at least one solution of the associated differential equation of the type (5.2) may be obtained in which such a series is defined. As the order of (5.2) increases, the algebraic complication of the working out becomes extremely marked.

REFERENCES

Ince, E. L. (1926). Ordinary Differential Equations. Longmans, Green & Co., New York.
Slater, L. J. (1966). Generalized Hypergeometric Functions. Cambridge University Press, London.