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SOME INTEGRALS AND SERIES RELATIONS FOR BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LEGENDRE POLYNOMIALS

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Recently Konhauser (1965, 1967) discussed various properties of biorthogonal polynomials and also introduced a biorthogonal pair of polynomials which is, in a certain sense suggested by the orthogonal set of the generalized Laguerre polynomials. Prabhakar and Tomar (1979) have given a biorthogonal pair of polynomial sets $\{U_n(x; k)\}$ and $\{V_n(x; k)\}$ which is analogously suggested by the Legendre polynomials. In the present paper we obtain some integrals and series relations for both the polynomial sets.

1. INTRODUCTION

Konhauser (1965) studied the general properties of biorthogonal polynomials and introduced (Konhauser 1967) polynomial sets $\{Y_n^{(\alpha)}(x; k)\}$ and $\{Z_n^{(\alpha)}(x; k)\}$ which form a biorthogonal pair with respect to the weight function and the interval of the generalized Laguerre polynomials; this pair was also studied by one of the present authors (see Prabhakar 1970, 1971). In a recent paper we have shown that sets $\{U_n(x; k)\}$ and $\{V_n(x; k)\}$ form a biorthognal pair with respect to the weight function and the interval of the Legendre polynomials $P_n(x)$ (see Prabhakar and Tomar 1979). Here $U_n(x; k)$ is a polynomial of degree n in the basic polynomial $\left(\frac{1-x}{2}\right)$ and $V_n(x; k)$ is a polynomial of degree n in the basic polynomial $\left(\frac{1-x}{2}\right)^k$, k being a positive integer. For $k = 1$, both $U_n(x; k)$ and $V_n(x; k)$ reduce to $P_n(x)$, and the biorthogonality condition reduces to the orthogonality condition satisfied by the Legendre polynomials.

To get the closed forms of biorthogonal polynomials is not at all simple. Konhauser (1967) could give the closed form for only the polynomial $Z_n^{(\alpha)}(x; k)$. Prabhakar and Tomar have been able to obtain the following closed forms for both the polynomials, namely $U_n(x; k)$ and $V_n(x; k)$:

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$$U_n(x;k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\left(\frac{j+1}{k}\right)_n}{\left(\frac{1}{k}\right)_n} \left(\frac{1-x}{2}\right)^j \qquad \text{...(1.1)}$$

$$V_n(x;k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(1+n)_{kj}}{(1)_{kj}} \left(\frac{1-x}{2}\right)^{kj}. \qquad \text{...(1.2)}$$

2. Integrals

In this section integrals involving the polynomials $U_n(x;k)$ and integrals involving $V_n(x;k)$ are evaluated.

(a) We now evaluate integrals involving $U_n(x;k)$. First, we show that

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v\, U_n(1-2x\sin^{2h}\theta); k)\, d\theta = \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)\, n!}{2^{v+1}\left(\frac{1}{k}\right)_n}$$

$$\times\, {}_2F_4^{(*)}\left[\begin{matrix}\left(\frac{1}{k}, n+\frac{1}{k}\right), (2h, v+1); \\ \left(h, \frac{v}{2}+u+1\right), \left(h, \frac{v}{2}-u+1\right), (-1, n+1), \left(\frac{1}{k}, \frac{1}{k}\right);\end{matrix} \frac{-x}{2^{2h}}\right] \qquad \text{...(2.1)}$$

and

$$\int_0^{\pi/2} \cos u\theta\, (\cos\theta)^v\, U_n((1-2x\cos^{2h}\theta); k)\, d\theta = \frac{\pi n!}{2^{v+1}\left(\frac{1}{k}\right)_n}$$

$$\times\, {}_2F_4^{(*)}\left[\begin{matrix}\left(\frac{1}{k}, n+\frac{1}{k}\right), (2h, v+1); \\ \left(h, \frac{v}{2}+\frac{1}{2}+1\right), \left(h, \frac{v}{2}-\frac{u}{2}+1\right), (-1, n+1), \left(\frac{1}{k}, \frac{1}{k}\right);\end{matrix} \frac{-x}{2^{2h}}\right] \qquad \text{...(2.2)}$$

where $u = 0, 1, 2, \ldots$ and $\operatorname{Re}(v) > 0$ and h is a positive integer. Here ${}_2F_4^{(*)}$ is Wright's generalized hypergeometric function [Erdélyi 1953 (4.1) and Wright 1940].

To prove (2.1), we use (1.1) in the integrand and change the order of integration and summation; the left-hand member then can be written as

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\left(\frac{j+1}{k}\right)_n}{\left(\frac{1}{k}\right)_n} x^j \int_0^{\pi/2} \cos 2u\theta \sin^{v+2hj}\theta \, d\theta. \qquad \ldots(2.3)$$

Now evaluating the integral with the help of the result [Erdélyi 1953, 1.5.1 (30)]

$$\int_0^{\pi/2} \cos 2u\theta \,(\sin\theta)^v \, d\theta = \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{v+1}\Gamma\left(\frac{\nu}{2}+\frac{u}{2}+1\right)\Gamma\left(\frac{\nu}{2}-\frac{u}{2}+1\right)}$$

where $u = 0, 1, 2, \ldots$ and $\mathrm{Re}\,(\nu) \geqslant 0$, we find that (2.3) is

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\left(\frac{j+1}{k}\right)_n}{\left(\frac{1}{k}\right)_n} \frac{\Gamma(\nu+1+2hj)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)\,x^j}{2^{v+1+2hj}\Gamma\left(\frac{\nu}{2}+u+1+hj\right)\Gamma\left(\frac{\nu}{2}-u+1+hj\right)}$$

which leads to (2.1).

For $k = 1$, $U_n(x; k)$ becomes $P_n(x)$ and (2.1) gives the following integral for Legendre polynomials:

$$\int_0^{\pi/2} \cos 2u\theta \,(\sin\theta)^v\, P_n((1-2x\sin^{2h}\theta); k)\, d\theta = \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{v+1}}$$

$$\times\ {}_2F_4^{(*)}\left[\begin{matrix}(1, n+1), (2h, \nu+1); \\ \left(h, \frac{\nu}{2}+u+1\right), \left(h, \frac{\nu}{2}-u+1\right); (-1, n+1), (1, 1);\end{matrix} \left(-\frac{x}{2^{2h}}\right)\right]. \qquad \ldots(2.4)$$

Integral (2.2) can be evaluated in a similar manner by using the formula [Erdélyi 1953, 1.5.1 (30)]:

$$\int_0^{\pi/2} \cos m\theta \,(\cos\theta)^n \, d\theta = \frac{\pi\Gamma(m+1)}{2^{n+1}\Gamma\left(\frac{n}{2}+\frac{m}{2}+1\right)\Gamma\left(\frac{n}{2}-\frac{m}{2}+1\right)}$$

where m is a positive integer.

We next give the following two integral formulae:

$$\int_0^{\pi/2} \cos(\alpha+\beta)\,\theta\,(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\, U_n((1-2x\tan^{2s}\theta); k)\, d\theta =$$

(*equation continued on p.* 866)

$$= \frac{n!\, 2^{\alpha-1}}{\Gamma(\alpha+\beta)\left(\frac{1}{k}\right)_n} {}_3F_3^{(*)}\left[\begin{matrix}\left(\frac{1}{k}, n+\frac{1}{k}\right), \left(\delta, \frac{\alpha}{2}\right), (-2\delta, \beta) \\ \left(\frac{1-\alpha}{2}, \delta\right)\left(\frac{1}{k}, \frac{1}{k}\right), (-1, n+1);\end{matrix} -2^{2\delta}x\right] \quad \ldots(2.5)$$

and

$$\int_0^{\pi/2} \sin(\alpha+\beta)\theta\,(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\, U_n((1-2x\tan^{2\delta}\theta); k)\, d\theta$$

$$= \frac{\sqrt{\pi}\; n!\, 2^{\alpha-1}}{\Gamma(\alpha+\beta)\left(\frac{1}{k}\right)_n} {}_3F_3^{(*)}\left[\begin{matrix}\left(\frac{1}{k}, n+\frac{1}{k}\right), \left(\delta, \frac{1+\alpha}{2}\right), (-2\delta, \beta); \\ (-1, n+1), \left(\frac{1}{k}, \frac{1}{k}\right), \left(1-\frac{\alpha}{2}, -\delta\right);\end{matrix} -2^{2\delta}x\right] \quad \ldots(2.6)$$

where δ is a positive integer, Re $(\alpha) > 0$ and Re $(\beta) > 0$.

These can be evaluated by using the results (MacRobert 1961)

$$\int_0^{\pi/2} \cos(\alpha+\beta)\theta\,(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\, d\theta = \frac{\sqrt{\pi}\, 2^{\alpha-1}\Gamma(\beta)\,\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha+\beta)}$$

and

$$\int_0^{\pi/2} \sin(\alpha+\beta)\theta\,(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\, d\theta = \frac{\sqrt{\pi}\, 2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma(\beta)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma(\alpha+\beta)}$$

where Re $(\alpha) \geqslant 0$ and Re $(\beta) \geqslant 0$.

(b) We now give corresponding integral formulae for the other biorthogonal polynomial, viz. $V_n(x; k)$:

$$\int_0^{\pi/2} \cos 2\theta\,(\sin\theta)^{v}\, V_n(1-2x\sin^{2h}\theta); k)\, d\theta$$

$$= \frac{\Gamma(v+1)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{v+1}\Gamma\left(\frac{v}{2}+u+1\right)\Gamma\left(\frac{v}{2}-u+1\right)}$$

$$\times\; {}_{1+2hk+k}F_{2hk+k}\left[\begin{matrix}-n, \triangle(k, 1+n), \triangle(2hk, v+1); \\ \triangle(k, 1), \triangle\left(hk, \frac{v}{2}+u+1\right), \triangle\left(hk, \frac{v}{2}-u+1\right);\end{matrix} x^k\right] \quad \ldots(2.7)$$

$$\int_0^{\pi/2} \cos u\theta \, (\cos \theta)^{v} \, V_n((1 - 2x \cos^{2h} \theta); k) \, d\theta$$

$$= \frac{\pi\Gamma(v+1)}{2^{v+1}\Gamma\left(\frac{v}{2} + \frac{u}{2} + 1\right)\Gamma\left(\frac{v}{2} - \frac{u}{2} + 1\right)}$$

$$\times \; {}_{1+k+2hk}F_{2hk+k}\left[\begin{matrix} -n, \triangle(k, n), \triangle(2hk, v+1); \\ \triangle(k, 1), \triangle\left(hk, \frac{v}{2} + \frac{u}{2} + 1\right), \triangle\left(hk, \frac{v}{2} - \frac{u}{2} + 1\right); \end{matrix} \; x^k\right] \qquad \ldots(2.8)$$

$$\int_0^{\pi/2} \cos (\alpha + \beta) \theta \, (\sin \theta)^{\alpha-1} (\cos \theta)^{\beta-1} \, V_n((1 - 2x \tan^{2\delta} \theta); k) \, d\theta$$

$$= \frac{\sqrt{\pi} \; 2^{\alpha-1}\Gamma(\beta)\,\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha+\beta)}$$

$$\times \; {}_{1+k+2\delta k}F_{k+2\delta k}\left[\begin{matrix} -n, \triangle(k, 1+n), \triangle\left(\delta k, \frac{1-\alpha}{2}\right), \triangle\left(\delta k, \frac{\alpha}{2}\right); \\ \triangle(k, 1), \triangle(2\delta k, 1-\beta); \end{matrix} \; -((-1)^{\delta} x)^k\right] \qquad \ldots(2.9)$$

and

$$\int_0^{\pi/2} \cos (\alpha + \beta) \theta \, (\sin \theta)^{\alpha-1} (\cos \theta)^{\beta-1} \, V_n((1 - 2x \tan^{2\delta} \theta); k) \, d\theta$$

$$= \frac{\sqrt{\pi} \; 2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma(\beta)}{\Gamma\left(1 - \frac{\alpha}{2}\right)\Gamma(\alpha+\beta)}$$

$$\times \; {}_{1+k+2\delta k}F_{k+2\delta k}\left[\begin{matrix} -n, \triangle(k, 1+n), \triangle\left(\delta k, \frac{\alpha}{2}\right), \triangle\left(\delta k, \frac{1+\alpha}{2}\right); \\ \triangle(k, 1), \triangle(2\delta k, 1-\beta); \end{matrix} \; ((-1)^{\delta} x)^k\right] \qquad \ldots(2.10)$$

where n, h, δ are positive integers, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\triangle(k, n)$ denote the set of 'k' parameters $\frac{n}{k}, \frac{n+1}{k}, \ldots, \frac{n+k-1}{k}$.

These integrals can also be evaluated in a similar manner as the results (2.1) to (2.6) remembering that

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},\ (a)_{kn} = k^{kn} \prod_{i=0}^{k-1} \left(\frac{a+i}{k}\right)_n \text{ and } (a)_{-n} = \frac{(-1)^n}{(1-\alpha)_n}.$$

3. Applications

The integral relations established in the previous section can be used to obtain corresponding series relations for the polynomials $U_n(x; k)$ and $V_n(x; k)$. For instance, using (2.1) we can prove that

$$(\sin\theta)^\nu\, U_n((1 - 2x\sin^{2h}\theta); k) = \frac{n!}{\pi 2^{\nu-1}\left(\frac{1}{k}\right)_n} \sum_{r=0}^{\infty} \Gamma(\tfrac{1}{2}+r)$$

$$\times\, \Gamma(\tfrac{1}{2}-r)\, {}_2F_4^{(*)}\left[\begin{matrix}\left(\frac{1}{k}, n+\frac{1}{k}\right), (2h, \nu+1); \\ \left(h, \frac{\nu}{2}+r+1\right), \left(h, \frac{\nu}{2}-r+1\right), (-1, n+1), \left(\frac{1}{k}, \frac{1}{k}\right);\end{matrix} -\frac{x}{2^{2h}}\right]$$

$$\times \cos 2r\theta \qquad \ldots(3.1)$$

where Re $(\nu) \geqslant 0$, h is a positive integer and $0 < \theta < \pi/2$.

If we denote $(\sin\theta)^\nu U_n((1 - 2x\sin^{2h}\theta); k)$ by $f(\theta)$, we can write

$$f(\theta) = (\sin\theta)^\nu U_n((1 - 2x\sin^{2h}\theta); k) = \sum_{r=0}^{\infty} A_r \cos 2r\theta,\ 0 < \theta < \pi/2.$$

Since $f(\theta)$ is continuous and of bounded variation in $(0, \pi/2)$, the coefficients A_r can be computed by the help of (2.1); we get

$$A_r = \frac{\Gamma(\frac{1}{2}+r)\,\Gamma(\frac{1}{2}-r)\, n!}{\pi 2^{\nu-1}\left(\frac{1}{k}\right)_n}$$

$$\times\, {}_2F_4^{(*)}\left[\begin{matrix}\left(\frac{1}{k}, n+\frac{1}{k}\right), (2h, \nu+1); \\ \left(h, \frac{\nu}{2}+r+1\right), \left(h, \frac{\nu}{2}-r+1\right), (-1, n+1), \left(\frac{1}{k}, \frac{1}{k}\right);\end{matrix} -\frac{x}{2^{2h}}\right]$$

The series relations corresponding to other integral formulae can be established on parallel lines.

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