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ON ORTHOGONAL POLYNOMIALS RELATED TO THE ULTRASPHERICAL POLYNOMIALS

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In the present note, we first connect the even and odd generalized ultraspherical polynomials of Thakare and Bhonsle (1974) with the Jacobi polynomials and then use these relationships to deduce certain properties of even and odd generalized ultraspherical polynomials from known results involving the Jacobi polynomials due to Srivastava (1969, 1972, 1974), Singhal and Srivastava (1972), and Srivastava and Lavoie (1975).

1. INTRODUCTION

In a recent paper, Thakare and Bhonsle (1974) studied the even and odd generalized ultraspherical polynomials which are orthogonal. They established a number of properties of these even and odd polynomials, such as Rodrigues formulae, contour integrals, recurrence relations and a generating function.

The Rodrigues formulae are

$$T_{2pk}(z) \equiv T_{2pk}(z, \alpha, \beta) = \frac{(-1)^p z^{\beta}(1-z)^{-\alpha}}{(1+\alpha)_p} D^p [z^{p-\beta}(1-z)^{\alpha+p}] \qquad \ldots(1.1)$$

and

$$T_{2pk+1}(z) \equiv T_{2pk+1}(z, \alpha, \beta) = \frac{(-1)^p (1-z)^{-\alpha}}{(1+\alpha)_p} D^p [z^{\beta+p}(1-z)^{\alpha+p}] \qquad \ldots(1.2)$$

where $\beta = 1/2k$.

It is interesting to observe that the so-called even and odd generalized ultraspherical polynomials (occurring in our results) are merely constant multiples of special Jacobi polynomials $P_n^{(\alpha-n,\beta-n)}(x)$. Specifically, we have the following connecting relationships:

$$T_{2pk}(z, \alpha, \beta) = \{p!/(1+\alpha)_p\} P_p^{(\alpha,-\beta)}(2z-1) \qquad \ldots(1.3)$$

$$T_{2pk+1}(z, \alpha, \beta) = \{p!/(1+\alpha)_p\} z^{\beta} P_p^{(\alpha,\beta)}(2z-1) \qquad \ldots(1.4)$$

where $\beta = 1/2k$.

Consequently, it would be straightforward to derive various results given earlier by Thakare and Bhonsle (1974) for these polynomials from the corresponding well-known results involving Jacobi polynomials.

The purpose of this note is to indicate the usefulness of relationships (1.3) and (1.4) in trivially deriving some more results, such as linear generating relations and bilateral generating relations for these polynomials.

2. Generating Relations

First we mention the following linear generating relations

$$\sum_{p=0}^{\infty} \frac{(1+\alpha-p)_p\,(\beta-\alpha-\nu)_p\,(\beta')_p}{(1+\alpha-\beta-p+\nu)_p\,(\alpha')_p} T_{2(p+\nu)k}(z;\alpha-p,\beta+p)\frac{t^p}{p!}$$

$$=\frac{(1+\alpha-\beta+\nu)_\nu}{(1+\alpha)_\nu}\,\frac{z^\beta}{(z-1)^{\beta-\nu}}$$

$$\times F_2\left[\beta-\alpha-\nu,\beta-\nu,\beta';\beta-\alpha-2\nu,\alpha';\frac{1}{1-z},zt\right] \qquad \ldots(2.1)$$

and

$$\sum_{p=0}^{\infty} \frac{(1+\alpha-p)_p\,(-\beta-\alpha-\nu)_p\,(\beta')_p}{(1+\alpha+\beta-p+\nu)_p\,(\alpha')_p} T_{2(p+\nu)k+1}(z;\alpha-p,\beta-p)\frac{t^p}{p!}$$

$$=\frac{(1+\alpha+\beta+\nu)_\nu}{(1+\alpha)_\nu}(z-1)^{\beta+\nu}$$

$$\times F_2\left[-\beta-\alpha-\nu,-\beta-\nu,\beta';-\beta-\alpha-2\nu,\alpha';\frac{1}{1-z},t\right] \qquad \ldots(2.2)$$

where $\nu = 0, 1, 2, \ldots$, and F_2 is one of the four Appell functions. The analogous generating relations for the Jacobi polynomials have been established by Srivastava [see, e.g., Srivastava (1974, 1972, 1969)].

Secondly, we state the following two theorems concerning a class of bilateral generating relations for these polynomials.

Theorem 1 — Let

$$F[z,t]=\sum_{p=0}^{\infty} A_p T_{2pk}(z;\alpha-p,\beta+p)\,t^p/p! \qquad \ldots(2.3)$$

where $A_p \neq 0$ are arbitrary constants. Then

$$(1 + zt)^{\alpha} (1 - t(1 - z))^{-\beta} F\left[z(1 - (1 - z)t), \frac{zt}{(1 - (1 - z)t)(1 + zt)}\right]$$

$$= \sum_{p=0}^{\infty} (1 + \alpha - p)_p T_{2pk}(z; \alpha - p, \beta + p) b_p(y) t^p \qquad ...(2.4)$$

where $b_p(y)$ is a polynomial of degree p in y, given by

$$b_p(y) = \sum_{r=0}^{p} \frac{A_r y^r}{(1 + \alpha - r)_r (p - r)! r!} \qquad ...(2.5)$$

Theorem 2 — If

$$F[z, t] = \sum_{p=0}^{\infty} A_p T_{2pk+1}(z; \alpha - p, \beta - p) t^p/p! \qquad ...(2.6)$$

then

$$(1 + t)^{\alpha} F\left[z - (1 - z)t, \frac{zt}{1 + t}\right] = \sum_{p=0}^{\infty} (1 + \alpha - p)_p$$

$$\times T_{2pk+1}(z; \alpha - p, \beta - p) b_p(y) t^p \qquad ...(2.7)$$

where $b_p(y)$ is given by (2.5).

In view of the relationships (1.3) and (1.4), the bilateral generating functions (2.4) and (2.7) can be deduced fairly easily by further specializing Corollary 6 of a theorem of Singhal and Srivastava (1972, p. 759). Indeed, more general bilateral generating functions can also be obtained by specializing a general theorem due to Srivastava and Lavoie [1975, p. 319, eqn. (107)].

3. Applications

We use Theorem 1 and the linear generating relation (2.1) with $\nu = 0$ to obtain a bilateral generating relation for the even polynomials.

Taking $A_p = \dfrac{(1 + \alpha - p)_p (\beta - \alpha)_p (\beta')_p}{(1 + \alpha - \beta - p)_p (\alpha')_p}$ in (2.3) we get from (2.4)

and (2.1) with $\nu = 0$

$$(1 + zt)^{\alpha} (1 - (1 - z)t - z^{-1})^{-\beta} F_2\Big[\beta - \alpha, \beta, \beta'; \beta - \alpha, \alpha';$$

$$\frac{1}{1 - z\{1 - (1 - z)t\}}, \frac{yzt}{1 + tz}\Big] =$$

(*equation continued on p.* 873)

$$= \sum_{p=0}^{\infty} (1 + \alpha - p)_p \, T_{2pk}(z; \alpha - p, \beta + p) \, {}_2F_1(-p, \beta'; \alpha'; y) \, t^p/p!. \qquad \ldots(3.1)$$

Similarly, for the odd polynomials, we have

$$(1 + t)^{\alpha} (z - (1 - z)\, t - 1)^{\beta} F_2\Big[-\beta - \alpha, -\beta, \beta'; -\beta - \alpha, \alpha';$$
$$\frac{1}{1 - \{z - (1 - z)\, t\}}, \frac{yt}{1 + t}\Big]$$
$$= \sum_{p=0}^{\infty} (1 + \alpha - p)_p \, T_{2pk+1}(z; \alpha - p, \beta - p)$$
$$\times \, {}_2F_1(-p, \beta'; \alpha'; y) \, t^p/p!. \qquad \ldots(3.2)$$

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