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A FUNDAMENTAL GROWTH RELATION BETWEEN THE SUPREMUM MODULUS AND THE MAXIMUM TERM OF AN ENTIRE DIRICHLET SERIES

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This paper considers certain asymptotic relations of the Valiron-Wiman theory concerning a general entire Dirichlet series. The main interest of this work is to consider the maximum term μ and the supremum modulus M (on the ordinates) of such a series and prove that ln M and ln μ have the same growth with respect to the Ritt family $\mathcal{R}$ under weakest possible conditions on the exponents of the series (without further restrictions on the Ritt order of the series).

1. INTRODUCTION

Let f be a non-constant entire Dirichlet series, specified by

$$f(z) = \sum_{n=0}^{+\infty} a_n e^{\lambda_n z}, \quad \forall\ z \in C,$$

where $a_n \in C, (n = 0, 1, 2, \ldots)$ and $\{\lambda_n : n = 0, 1, 2, \ldots\}$ is a strictly increasing sequence in $\mathbf{R}$ tending to $+\infty$, with

$$\limsup_{n \to +\infty} \frac{\ln n}{\ln \lambda_n} < +\infty.$$

Let $M = M_f$, $\mu = \mu_f$ and $\Delta = \Delta_f$, respectively, be the supremum modulus, the maximum term and the central index of f, defined by

$$M(x) = \sup\left[\,|f(x+iy)| : y \in \mathbf{R}\right],$$

$$\mu(x) = \max\left[\,|a_n|\, e^{\lambda_n x} : n = 0, 1, 2, \ldots\right]$$

and

$$\Delta(x) = \max\left[\lambda_n : \mu(x) = |a_n|\, e^{\lambda_n x}\right]$$

for every x in $\mathbf{R}$.

The extensive Valiron-Wiman theory [see e.g. Valiron (1949), Gopala Krishna (1965, 1969 and 1970) and Gopala Krishna and Nagaraja Rao (1977)] concerning

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power series may be considered as studying f in the special case when $\lambda_n = n$ for $n = 0, 1, 2, \ldots$. Many attempts have been made to extend the theory to the case of an entire Dirichlet series (see e.g. the rest of the works under reference not referred so far).

Ritt (1928) (among others) introduced the notion, now known as the Ritt order $\rho = \rho_R$ of f, given by

$$\rho = \limsup_{x \to +\infty} \frac{\ln \ln M(x)}{x}.$$

This along with other notions such as the Ritt type, the Ritt lower order of f specify 'the growth' or the asymptotic behaviour of $\ln M$ at $+\infty$ w.r.t. the Ritt family $\mathcal{R}$ consisting of all real valued functions g, whose domain is an interval of $\mathbf{R}$ not bounded above and which is such that $g(x) = te^{\alpha x}$ as $x \to +\infty$, for some positive α and t.

Following Gopala Krishna and Nagaraja Rao (1977) (based on various considerations discussed by them in detail) we understand by the $\mathcal{R}$-growth of a nonnegative real function h whose domain is an interval of $\mathbf{R}$ not bounded above, as the ordered pair $(\bar{U}, \underline{U})$, defined as follows : $\bar{U}$ is the set of all $g \in \mathcal{R}$ such that $g(x) \geqslant \theta\, h(x)$ as $x \to +\infty$ for some $\theta \in (1, +\infty)$ and $\underline{U}$ is the set of all $g \in \mathcal{R}$ such that $g(x) \leqslant \theta h(x)$ as $x \to +\infty$ for some $\theta \in (0, 1)$.

The main concern of this work is to establish the following:

Theorem 1.1 — $\ln M$ and $\ln \mu$ have the same $\mathcal{R}$-growth.

This is an extension, as detailed earlier, of its analogue for the case of power series [Gopala Krishna (1965, Lemma 8)], which served a basic role*. Following Gopala Krishna and Nagaraja Rao (1977) we prove the analogous result for Dirichlet series in one complex variable. Accordingly we need certain other results, which may be of interest in themselves, such as the fact that $\ln M(x) \sim \ln \mu(x)$ as $x \to +\infty$ when f is of finite Ritt order (Sugimura 1928–29). But we shall detail out such results in the course of our discussions as and when they become relevant.

2. Preliminary Results

In the first place we need the following:

Proposition 2.1 — $\ln M(x) \sim \ln \mu(x)$ as $x \to +\infty$ outside an exceptional set of finite Lebesgue measure in $(0, +\infty)$.

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*Analogously Th. (1.1) includes the fact that the Ritt order, Ritt type, Ritt lower order etc. of f may be defined, equivalently, using μ in the place of M.

PROOF : This is an easy consequence of Theorem $I(a)$ of Seremeta (1978). [It may however be noted that 'finite' in the Russian original, in connection with the measure under consideration, is translated as 'positive'.]

Proposition 2.2 — Let $h \in \mathfrak{R}$. Let $\{x_n : n = 0, 1, 2, ...\}$ be an unbounded increasing sequence in the domain I of h and let $\theta \in (0, 1)$. Then the set

$$E = [x \in I : x \leqslant x_n, \theta h(x_n) \leqslant h(x) \text{ for some } n \geqslant 0]$$

has infinite Lebesgue measure.

PROOF : The result is analogous to Lemma (6.4) of Gopala Krishna and Nagaraja Rao (1977) and its proof is similar to the same.

3. PROOF OF THEOREM 1.1

Let $h \in \mathfrak{R}$ and let

$$\begin{matrix} c \\ c' \end{matrix} = \lim_{x \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\ln M(x)}{h(x)}$$

and

$$\begin{matrix} d \\ d' \end{matrix} = \lim_{x \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\ln \mu(x)}{h(x)}.$$

In order to prove the theorem it is sufficient to prove that $c = d$ and $c' = d'$.

If d is finite then it follows by a known result [see, Rajagopal and Reddy (1965–66, Theorem 1)] that the Ritt order of f is finite. Hence, if one of c and d is finite, by a known theorem (Sugimura 1928–29) we have $\ln M \sim \ln \mu$. Thus $c = d$ in any case.

Now $d' \leqslant c'$ is a consequence of the fact that $\mu \leqslant M$, which is an easy consequence of the Hadamard formula for coefficients of Dirichlet series [see Sansone and Gerretsen (1960, p. 386)]. If $d' = +\infty$ we get $d' = c'$. Let us assume that $d' < +\infty$ and let $d' < q < +\infty$. Now by Proposition (2.2) and by Lemma (6.3) of Gopala Krishna and Nagaraja Rao (1977) we have that the set

$$S = [x \in I : \ln \mu(x) \leqslant qh(x)]$$

has infinite Lebesgue measure.

We, therefore, observe (using Prop. 2.1) that there exists an exceptional set C of finite Lebesgue measure such that, S-C contains a sequence $\{x_n : n = 0, 1, 2, ...\}$ increasing to $+\infty$ for which

$$\ln \mu(x_n) \leqslant qh(x_n) \text{ and } \ln M(x_n) \sim \ln \mu(x_n) \text{ as } n \to +\infty.$$

Thus $c' \leqslant q$, which is true for every $q > d'$. Hence $c' \leqslant d'$. This, together with the fact that $d' \leqslant c'$, implies that $c' = d'$, and this completes the proof of the theorem.

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