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PROBLEM FOR PROPAGATION OF WAVES IN AN ELASTIC SOLID MEDIUM WITH THE BOUNDARY CONDITIONS AS DISTRIBUTIONS

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In this paper the authors solve the problem for propagation of waves in the elastic solid medium. For this purpose, a semi-infinite rectangular parallelopiped bar with boundary conditions as distributions is considered. The problem is solved by the application of distributional Laplace finite double Fourier sine transformation.

1. INTRODUCTION

Waves in a continuous and infinite elastic medium may result from imposing certain initial or boundary conditions of displacement or by application of forces on the medium with disturbances propagating into undisturbed medium. In many problems there may be singularities in the initial and boundary conditions. Such problems cannot be solved with the help of conventional conditions. These problems can be correctly specified with the help of generalized functions, of which the distributions are particular cases.

Here, in this paper, we shall consider the wave propagation in an elastic rectangular parallelopiped bar by taking the origin at one end of the bar and the axes of X, Y, Z along the coterminus edges through the origin, Z-axis being taken as the vertical axis. The length and breadth each are taken to be equal to π along the X and Y axes respectively. Let the bar be of infinite extent in the z-direction.

We shall fix the faces (Carrier and Pearson 1976) as follows:

(i) $x = 0,\ 0 \leqslant y \leqslant \pi,\ z \geqslant 0$

(ii) $x = \pi,\ 0 \leqslant y \leqslant \pi,\ z \geqslant 0$

(iii) $y = 0,\ 0 \leqslant x \leqslant \pi,\ z \geqslant 0$

(iv) $y = \pi,\ 0 \leqslant x \leqslant \pi,\ z \geqslant 0$

so that on these faces no displacement or inclination at any time during the motion are possible.

Let the medium be initially at rest and $U(t, x, y, z) = D_t\, U(t, x, y, z) = 0$ where $U(t, x, y, z)$ is the wave function satisfying the differential equation (Love 1952)

$$\frac{\partial^2 U}{\partial t^2} = c_1^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \qquad \ldots(1.1)$$

on the domain

$$R = \{(x, y, z, t) \mid 0 \leqslant x \leqslant \pi, 0 \leqslant y \leqslant \pi, z \geqslant 0, -\infty < t < \infty\}$$

where c_1 is the speed of the wave and is taken to be constant.

We shall assume that $U(t, x, y, z) \equiv 0$ on $-\infty < t < 0$. We shall apply an oscillatory forcing function on the surface $z = 0$ to begin the motion. We shall solve this problem of propagation of waves in the elastic rectangular parallelopiped bar having two of the boundary conditions as distributions by the use of the Laplace-finite double Fourier sine transformation of distributions.

2. Definitions, Terminology and Results

We shall give some definitions, terminology and the results used by Bhonsle and Subramanian (1979a, b). We shall denote the open set

$(-\infty, \infty) \times (0, \pi) \times (0, \pi)$ by I.

For each pair of real numbers a and b, the function $P_{a,b}(t)$ is defined as follows:

$$P_{a,b}(t) = e^{at},\ 0 \leqslant t < \infty$$
$$= e^{bt},\ -\infty < t < 0.$$

The space $LS_{a,b}(I)$ is defined as the space of all complex valued functions $\phi(t, x, y)$ that are infinitely differentiable with respect to t, x, y on I, and for each pair of non-negative integers k_1, k_2, the functionals

$$\rho_{a,b,k_1,k_2}\phi(t, x, y) = \sup_{\substack{-\infty<t<\infty \\ 0<x<\pi \\ 0<y<\pi}} | P_{a,b}(t)\, D_t^{k_1}\, \Omega_{x,y}^{k_2}\, \phi(t, x, y) | \text{ exist} \qquad \ldots(2.1)$$

where $\Omega_{x,y}^{k_2} \equiv (D_x^2 + D_y^2)^{k_2} = \sum_{p=0}^{k_2} \binom{k_2}{p} D_x^{2k_2-2p}\, D_y^{2p}$

and $D_t \equiv \frac{\partial}{\partial t},\ D_x \equiv \frac{\partial}{\partial x},\ D_y \equiv \frac{\partial}{\partial y}.$

$LS_{a,b}(I)$ is complete countable multinormed space. $LS_{a,b}(I)$ is a testing function space.

Let w denote either a finite real number or $-\infty$, and z denote either a finite real number or $+\infty$. We choose two monotonic sequences $\{a_p\}_{p=1}^{\infty}$ and $\{b_p\}_{p=1}^{\infty}$ such that $a_p \to w+$ and $b_p \to z-$. Then

$LS(w, z) = \bigcup_{p=1}^{\infty} LS_{a_p,b_p}$ is defined as the countable union space.

The dual space $LS'_{a,b}(I)$ of $LS_{a,b}(I)$ is the collection of all continuous linear functionals on $LS_{a,b}(I)$. $LS'_{a,b}(I)$ is also complete. $LS'(w, z)$ is the dual space of $LS(w, z)$ and is also complete.

Distributional Laplace Finite Double Fourier Sine Transformation

For a given Laplace finite double Fourier sine transformable function

$$f \in LS'(w, z)$$

if D_f denotes the strip of definition given by

$$D_f = \{(s, m, n) \mid p_f < R_e s < q_f, \; m \text{ and } n \text{ are positive integers}\}$$

where

$$p_f = \inf \{w \mid f \in LS'(w, z)\}$$

$$q_f = \sup \{z \mid f \in LS'(w, z)\}$$

then the Laplace finite double Fourier sine transform $F(s, m, n)$ of $f(t, x, y)$ is defined by

$$\mathcal{Z}\mathcal{F}[f(t, x, y)] \stackrel{\Delta}{=} F(s, m, n)$$

$$\stackrel{\Delta}{=} \langle f(t, x, y), e^{-st} \sin mx \sin ny \rangle. \qquad \ldots(2.2)$$

That is, it is defined as the application of $f \in LS'(p_f, q_f)$ to the kernel

$$e^{-st} \sin mx \sin ny \in LS(p_f, q_f)$$

or equivalently as the application of $f \in LS'(I)$ to $e^{-st} \sin mx \sin ny \in LS_{a,b}(I)$ for any $p_f < a \leqslant R_e s \leqslant b < q_f$ and for positive integers m and n.

If $f(t, x, y)$ be a locally integrable function such that

$$e^{-st} \sin mx \sin ny \, f(t, x, y)$$

be absolutely integrable on I, then its conventional Laplace finite double Fourier sine transform

$$\int_0^\pi \int_0^\pi \int_{-\infty}^\infty f(t, x, y)\, e^{-st} \sin mx \sin ny \, dt\, dx\, dy \qquad \text{...(2.3)}$$

exists for at least $p_f < R_e s < q_f$ and for positive integers m and n. Further, this can be identified with our distributional Laplace finite double Fourier sine transformation (2.2). That is

$$\mathcal{Z}\mathcal{F}\,[f(t, x, y)] \triangleq \langle f(t, x, y), e^{-st} \sin mx \sin ny \rangle$$

$$= \int_0^\pi \int_0^\pi \int_{-\infty}^\infty f(t, x, y)\, e^{-st} \sin mx \sin ny \, dt\, dx\, dy. \qquad \text{...(2.4)}$$

Inversion Theorem

Let $f(t, x, y)$ be a Laplace finite double Fourier sine transformable function and $F(s, m, n)$ be the distributional Laplace finite double Fourier sine transform of $f(t, x, y)$ as defined by (2.4), then in the sense of convergence in $D'(I)$

$$f(t, x, y) = \lim_{R, M, N \to \infty} \left[\frac{1}{2\pi i} \cdot \frac{4}{\pi^2} \sum_{m=1}^{M} \sum_{n=1}^{N} \sin my \sin ny \right.$$

$$\left. \times \int_{\sigma - iR}^{\sigma + iR} e^{st} F(s, m, n)\, ds \right] \qquad \text{...(2.5)}$$

where σ is any fixed number in $p_f < \sigma < q_f$.

3. Statement of the Problem

We wish to find a conventional function $U(t, x, y, z)$ that satisfies the differential equation

$$[C^2 D_t^2 - (D_x^2 + D_y^2)]\, U(t, x, y, z)$$

$$= D_z^2\, U(t, x, y, z) \qquad \text{...(3.1)}$$

on the domain

$$R = \{(x, y, z, t) \mid 0 \leqslant x \leqslant \pi, 0 \leqslant y \leqslant \pi, z \geqslant 0, -\infty < t < \infty\}$$

with the boundary conditions:

(i) As $z \to 0^+$, $U(t, x, y, z)$ converges in the sense of $D'(I)$ to a distribution $f(t, x, y) \in LS'(w, z)$;

(ii) As $z \to \infty^+$, $U(t, x, y, z)$ converges in the sense of $D'(I)$ to zero.

Solution of the Problem

Now eqn. (3.1) can be written as

$$[C^2 D_t^2 - \Omega_{x,y}]\, U = D_z^2\, U. \qquad \text{...(3.2)}$$

By applying the distributional Laplace finite double Fourier sine transform $\mathcal{Z}\mathcal{F}$ to (3.4) and formally interchanging $\mathcal{Z}\mathcal{F}$ with D_z^2, we convert (3.2) into

$$[D_z^2 - (c^2 s^2 + m^2 + n^2)]\, \bar{U} = 0$$

where $\bar{U} = \mathcal{Z}\mathcal{F}(U)$

$$= \langle\, U(t, x, y, z), s^{-st} \sin mx \sin ny \,\rangle.$$

Solving this equation we get

$$\bar{U}(z, s, m, n) = A(s, m, n)\, e^{-pz} + B(s, m, n)\, e^{pz} \qquad \text{...(3.3)}$$

where $A(s, m, n)$ and $B(s, m, n)$ are constants that do not depend upon z, and

$$p^2 = c^2 s^2 + m^2 + n^2.$$

In view of the boundary condition (ii) and noting that

$$\lim_{z \to \infty^+} \bar{U}(z, s, m, n) = 0,$$

it is reasonable to choose $B(s, m, n) = 0$. Now matching upon the boundary condition (i) and noting that $\lim_{z \to 0^+} \bar{U}(z, s, m, n) = F(s, m, n)$ we obtain

$$A(s, m, n) = F(s, m, n)$$

where $F(s, m, n) = \langle\, f(t, x, y), e^{-st} \sin mx \sin ny \,\rangle.$

Therefore, we have from (3.3),

$$\bar{U}(z, s, m, n) = F(s, m, n)\, e^{-pz}. \qquad \text{...(3.4)}$$

Applying inversion theorem to the above equation we get

$$U(t, x, y, z) = \lim_{R, M, N \to \infty} \left[\frac{1}{2\pi i} \frac{4}{\pi^2} \sum_{m=1}^{M} \sum_{n=1}^{N} \sin mx \sin ny \times \int_{\sigma - iR}^{\sigma + iR} e^{st}\, F(s, m, n)\, e^{-pz}\, ds \right] \text{ in } D'(I).$$

For each $\phi(t, x, y) \in D(I)$, it can be shown that

$$\langle U(x, y, z, t), \phi(t, x, y) \rangle$$

$$= \int_0^\pi \int_0^\pi \int_{-\infty}^{\infty} \left[\frac{1}{2\pi i} \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin mx \sin ny \right.$$

$$\left. \times \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} F(s, m, n) e^{-\nu z} ds \right] \phi(t, x, y) \, dt \, dx \, dy$$

so that $U(t, x, y, z)$, as a conventional function is obtained as follows:

$$U(t, x, y, z) = \frac{1}{2\pi i} \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin mx \sin ny \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} F(s, m, n) e^{-\nu z} ds. \quad ...(3.5)$$

We now verify that (3.5) is truely our solution to (3.1).

Now, $F(s, m, n)$ is bounded on the domain R. The factor $e^{-\nu z}$ ensures that (3.5) converges to zero on the domain R for $Z \leqslant z < \infty$ $(Z > 0)$. Moreover, the series in (3.5) converges uniformly on the domain R. It can also be seen that the series obtained by applying the operators $C^2 D_t^2$, $\Omega_{x,y}$ and D_z^2 separately under the summation and integral signs converge uniformly on R.

Thus, applying the operator $\{D_z^2 - [C^2 D_t^2 - \Omega_{x,y}]\}$ term by term to (3.5) we can see that it satisfies the differential equation (3.1) in the conventional sense.

To verify the boundary condition (i), we shall show that for each $\phi \in D(I)$, $U \in LS'(w, z)$,

$$\langle U(t, x, y, z), \phi(t, x, y) \rangle \to \langle f, \phi \rangle \text{ as } z \to 0^+. \quad ...(3.6)$$

To do this, we note that

$$\langle U(t, x, y, z), \phi(t, x, y) \rangle$$

$$= \lim_{z \to 0^+} \int_0^\pi \int_0^\pi \int_{-\infty}^{\infty} \frac{1}{2\pi i} \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin mx \sin ny$$

$$\times \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} F(s, m, n) e^{-(c^2 s^2 + m^2 + n^2) z} \phi(t, x, y) \, ds \, dt \, dx \, dy \quad ...(3.7)$$

The boundedness of $F(s, m, n)$ and the uniform convergence of the series allows us to take the limit under the integral and summation signs and we obtain

$$= \int_0^\pi \int_0^\pi \int_{-\infty}^{\infty} \frac{1}{2\pi i} \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin mx \sin ny$$

$$\times \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} F(s, m, n) \phi(t, x, y) \, ds \, dt \, dx \, dy$$

and on account of inversion theorem this expression equals $\langle f(t, x, y), \phi(t, x, y) \rangle$. This proves the boundary condition (i).

In an exactly similar manner, the boundary condition (ii) can be verified.

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