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ON THE COMPLETENESS OF GENERALIZED EIGEN FUNCTION EXPANSION ASSOCIATED WITH BESSEL FUNCTIONS WITH RESPECT TO ORDER

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The completeness of generalized eigen functions for a fourth order differential equation associated with Bessel's function and its derivative has been discussed.

§1. Solution of boundary value problems of mathematical physics is based on eigen function expansions. In treating these, problems of Sturm-Liouville type of second order differential equation play an important role. There we have an infinite set of eigenvalues with corresponding eigen functions with an orthogonal property. This enables us to represent an arbitrary function in terms of these. The limitation of the case is that at an end point of the basic interval either only the function or its derivative or a linear combination of these only can be made to vanish. But boundary value problems do arise with other complex form of end conditions where exact form of solution of the problem is difficult to obtain. We believe that one of the impediments in obtaining these is the lack of knowledge of functions satisfying the desired boundary conditions. The need and advantage of these functions is felt by us in studying the exact form of solution of a problem in cylindrical wedge. Nariboli (1965a, b) discussed the similar problem when he considered exact form of solution of elastostatic problems in beams and strips with different end conditions. The differential equation solved by him was of order four and its solutions are associated with cosine and sine functions.

In the present paper, we study the completeness of the expansions of two functions, not being independent of each other in terms of eigen functions of a pair of second order differential equation with end conditions involving the eigenvalues implicitly. These two second order differential equations are together equivalent to a single fourth order equation with complex form of end conditions. Application of the method is ready in elastostatic problem of wedges bounded by planes and circular cylinder.

§2. Consider the following pair of differential equations:

$$Lu = 2kV; \quad LV = 0 \qquad \ldots(2.1)$$

for $0 \leqslant r \leqslant a$, where the operator L is defined as

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(k^2 + \frac{4\nu^2}{r^2}\right)$$

and k is a real constant. The boundary conditions are that u, V are both finite for $r \to 0$ and for $r = a$

$$\frac{1}{r}\frac{du}{dr} = kV \qquad ...(2.2)$$

$$u = 0. \qquad ...(2.3)$$

The above statement poses an eigenvalue problem. Here the eigenvalue ν, a complex number, implicitly occurs inside the boundary condition (2.2). As a result, the eigenvalue problem is not a self-adjoint type. However, the eigen functions associated with the above problem are obtained as

$$u = \mathcal{E}_{2\nu}(kr) = rI'_{2\nu}(kr) - \frac{4\nu^2}{k^2 a}\frac{I_{2\nu}(ka)}{I'_{2\nu}(ka)} I_{2\nu}(kr) \qquad ...(2.4)$$

and $$V = I_{2\nu}(kr) \qquad ...(2.5)$$

and the corresponding eigenvalues ν_j, $j = 1, 2, ...$ are those roots of the equation

$$\mathcal{E}_{2\nu}(ka) = 0 \qquad ...(2.6)$$

for which $\text{Re}\,\nu > 0$. Naturally, the eigenvalues ν_j, $j = 1, 2, ...$ are the roots of the equation

$$I_{2\nu-1}(ka) = 0, \quad \text{where} \quad 0 < \text{Re}\,\nu < \tfrac{1}{2}. \qquad ...(2.7)$$

We can deduce the orthogonality relation

$$\int_0^a [\mathcal{E}_{2\nu_i}(kr)\, I_{2\nu_j}(kr) + \mathcal{E}_{2\nu_j}(kr)\, I_{2\nu_i}(kr)]\frac{dr}{r} = 0 \qquad ...(2.8)$$

for $i \neq j$, since $\mathcal{E}_{2\nu}(kr)$ satisfies the differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \left(k^2 + \frac{4\nu^2}{r^2}\right)u = 2kI_{2\nu}(kr).$$

We have

$$\int_0^a \frac{d}{dr}\left[r\frac{d}{dr}\mathcal{E}_{2\nu_i}(kr)\right] I_{2\nu_j}(kr)\, dr$$

$$= \int_0^a \frac{d}{dr}\left[r\frac{d}{dr} I_{2\nu_j}(kr)\right]\mathcal{E}_{2\nu_i}(kr)\, dr + ka^2 I_{2\nu_i}(ka)\, I_{2\nu_j}(ka); \; \text{Re}\,\nu > 0.$$

Using the differential equation for $\mathcal{E}_{2v}(kr)$, we get

$$4(\nu_i^2 - \nu_j^2) \int_0^a \mathcal{E}_{2v_i}(kr)\, I_{2v_j}(kr) \frac{dr}{r} + 2k \int_0^a I_{2v_i}(kr)\, I_{2v_j}(kr)\, dr$$

$$= ka^2 I_{2v_i}(ka)\, I_{2v_j}(ka)$$

and

$$4(\nu_j^2 - \nu_i^2) \int_0^a \mathcal{E}_{2v_j}(kr)\, I_{2v_i}(kr) \frac{dr}{r} + 2k \int_0^a I_{2v_i}(kr)\, I_{2v_j}(kr)\, dr$$

$$= ka^2 I_{2v_i}(ka)\, I_{2v_j}(ka)$$

by interchange of i, j. On subtracting the two final results we have the orthogonality relation (2.8).

Having this orthogonality relation at hand it enables us to proceed to the following expansion problem of two associated functions $F(r)$ and $G(r)$ in terms of the eigen functions:

$$F(r) = \sum_{\nu_j} a_v \mathcal{E}_{2v}(kr), \qquad \text{...(2.9)}$$

$$G(r) = \sum_{\nu_j} a_v I_{2v}(kr). \qquad \text{...(2.10)}$$

Using (2.8) in (2.9) and (2.10) we get

$$a_{v_j} = \frac{1}{K_{v_j}} \int_0^a [F(r)\, I_{2v_j}(kr) + G(r)\, \mathcal{E}_{2v_j}(kr)] \frac{dr}{r} \qquad \text{...(2.11)}$$

where the ortho-normalizing factor K_{v_j} is given by

$$K_{v_j} = 2 \int_0^a \mathcal{E}_{2v_j}(kr)\, I_{2v_j}(kr) \frac{dr}{r} \qquad \text{...(2.12)}$$

$$= \frac{1}{k} I^2_{2v_j}(ka) - (8\nu_j^2/k^2 a) \frac{I_{2v_j}(ka)}{I'_{2v_j}(ka)} \int_0^a \frac{I^2_{2v_j}(kr)}{r}\, dr.$$

But $$\int_0^a I^2_{2v_j}(kr) \frac{dr}{r} = -\frac{ak}{8\nu_j} I'^2_{2v}(ka) \frac{d}{dv} \left[\frac{I_{2v}(ka)}{I'_{2v}(ka)}\right]\Bigg|_{v=v_j}$$

[cf. Dougall (1899-1900)]. Hence, using $\mathcal{E}_{2v_j}(ka) = 0$, we get

$$K_{v_j} = \frac{I'^2_{2v_j}(ka)}{2k}\left[\frac{k^2a^2}{2v^2} + v\frac{d}{dv}\left\{\frac{I^2_{2v}(ka)}{I'^2_{2v}(ka)}\right\}\right]\Bigg|_{v=v_j}. \qquad ...(2.13)$$

Further, from (2.4), we have

$$\frac{d}{dv}[\mathcal{E}_{2v}(ka)]\Big|_{v=v_j} = aI'_{2v}(ka)\frac{d}{dv}\left\{1 - \frac{4v^2}{k^2a^2}\frac{I^2_{2v}(ka)}{I'^2_{2v}(ka)}\right\}\Bigg|_{v=v_j}.$$

Again using the result $\mathcal{E}_{2v_j}(ka) = 0$ in the above and after a little simplification we get

$$\left[\frac{d}{dv}\mathcal{E}_{2v}(ka)\right]_{v=v_j} = -\frac{4v_j}{k^2a}I'_{2v_j}(ka)\left[\frac{k^2a^2}{2v^2} + v\frac{d}{dv}\left\{\frac{I^2_{2v}(ka)}{I'^2_{2v}(ka)}\right\}\right]_{v=v_j}. \qquad ...(2.14)$$

Hence from (2.13) and (2.14) we get

$$K_{v_j} = -\frac{ka}{8v_j}I'_{2v_j}(ka)\left[\frac{d}{dv}\mathcal{E}_{2v}(ka)\right]_{v=v_j}. \qquad ...(2.15)$$

Thus, using the results of (2.11) and (2.15), the expansions in (2.9) and (2.10) can be rewritten as

$$F(r) = -\sum_{v_j}\frac{8v\mathcal{E}_{2v}(kr)}{kaI'_{2v}(ka)\dfrac{d}{dv}\mathcal{E}_{2v}(ka)}$$

$$\times\left[\int_0^a\{F(t)\,I_{2v}(kt) + G(t)\,\mathcal{E}_{2v}(kt)\}\frac{dt}{t}\right] \qquad ...(2.16)$$

and

$$G(r) = -\sum_{v_j}\frac{8vI_{2v}(kr)}{kaI'_{2v}(ka)\dfrac{d}{dv}\mathcal{E}_{2v}(ka)}$$

$$\times\left[\int_0^a\{F(t)\,I_{2v}(kt) + G(t)\,\mathcal{E}_{2v}(kt)\}\frac{dt}{t}\right]. \qquad ...(2.17)$$

The function $I_{2v-1}(ka)$ has no zeros in the half plane Re $v > \frac{1}{2}$ [cf. Dougall (1899-1900)] and so the above summations extend over those zeros of $I_{2v-1}(ka)$ and therefore of

$\mathcal{E}_{2v}(ka)$, for which $0 < \text{Re } \nu < \frac{1}{2}$. For $|\nu| \gg 1$, the asymptotic estimates of $I_{2v}(kr)$ and $I'_{2v}(ka)$ [cf. Naylor (1964)] are

$$I_{2v}(kr) \sim \frac{1}{\Gamma(2\nu+1)} \left(\frac{kr}{2}\right)^{2v},$$

$$I'_{2v}(ka) \sim \frac{1}{\Gamma(2\nu)} \left(\frac{ak}{2}\right)^{2v-1}.$$

Therefore, from (2.4) and from above we deduce the following asymptotic estimates of $\mathcal{E}_{2v}(kr)$ and $\mathcal{E}_{2v}(ka)$ as

$$\mathcal{E}_{2v}(kr) \sim \frac{1}{\Gamma(2\nu+2)} \left(\frac{kr}{2}\right)^{2v} \left[1 + \left(\frac{r}{a}\right)^{2v}\right] \times \frac{ka^2}{2}$$

$$\mathcal{E}_{2v}(ka) \sim \frac{2a}{\Gamma(2\nu+2)} \left(\frac{ak}{2}\right)^{2v+1}. \qquad \text{...(2.18)}$$

Using the estimates in (2.18), (2.17) has the integral representation

$$G(r) = -\frac{4}{2\pi i} \int_L \frac{I_{2v}(kr)}{I_{2v}(ka)\, \mathcal{E}_{2v}(ka)} \left[\int_0^a \{F(t)\, I_{2v}(kt) + G(t)\, \mathcal{E}_{2v}(kt)\} \frac{dt}{t} \right] d\nu \qquad \text{...(2.19)}$$

where L is the path parallel to the imaginary ν-axis and is so situated that on the path $0 < \text{Re } \nu < \epsilon$, ϵ being a small positive number.

The finite Lebedev transform deduced by Naylor (1963) is given by

$$F(\nu) = \int_0^a [I_v(ka)\, K_v(kr) - I_v(kr)\, K_v(ka)]\, f(r) \frac{dr}{r},$$

$$f(r) = \frac{1}{\pi i} \int_L \frac{I_v(kr)}{I_v(ka)} \nu F(\nu)\, d\nu. \qquad \text{...(2.20)}$$

At this stage we assume that the function $G(r)$ is of such nature that the transform pair (2.20) is applicable to (2.19). Then from (2.19) by differentiation and inversion with the help of (2.20) we get

$$G'(r) = \frac{8}{2\pi i} \int_L \frac{\nu I'_{2v}(kr)}{I_{2v}(ka)} \left[\int_0^a G(t)\, \{I_{2v}(ka)\, K_{2v}(kt) - I_{2v}(kt)\, K_{2v}(ka)\} \frac{1}{t}\, dt \right] d\nu. \qquad \text{...(2.21)}$$

The "formal" validity of (2.21) follows in view of

$$\delta(r-t)=\frac{4}{\pi i}\int_L \frac{I_{2v}(kr)}{I_{2v}(ka)}\left[I_{2v}(ka)\,K_{2v}(kr)-I_{2v}(kr)\,K_{2v}(ka)\right]\frac{v}{t}\,dv \qquad \ldots(2.22)$$

on simplifying the right-hand side of (2.21). Thus the validity of the expansion formula (2.17) i.e. its completeness is proved. As $F(r)$ and $G(r)$ are not independent the completeness of the other expansion in (2.16) automatically follows. Finally it is worth-mentioning that such an expansion theorem has application in elasto-static problem of an infinite cylindrical wedge when two rotation components are prescribed on plane boundaries and the cylindrical boundary being rigidly fixed with respect to rotation.

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