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FIXED POINT THEOREMS FOR MULTI-MAPPINGS

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In this paper, some fixed point theorems are proved for multi-mappings defined on uniform spaces. These employ suitable continuity conditions and require certain sets to be non-empty. Some well-known results for point-valued mappings including Banach's contraction principle are contained as special cases of the results obtained here.

1. INTRODUCTION

A multi-mapping T on a set X is a correspondence such that $T(x)$ is a subset of X for each $x \in X$, and a fixed point of T is a point x satisfying $x \in T(x)$. In this paper, we extend the principal fixed point result of Wong (1973, Theorems 2.1 and 2.4) and a result of Tarafdar (1974, Corollary 2.3) for point-valued mappings on uniform spaces to multi-mappings.

2. FIXED POINTS IN UNIFORM SPACES

Let $(X, \mathcal{U})$ be a uniform space. For the terminology of uniform spaces, we refer the reader to Kelley (1955). Let $\mathcal{B}$ denote the base for $\mathcal{U}$ consisting of all closed symmetric entourages. Let 2^X (resp. Cpt. (X)) denote the family of nonempty closed (resp. compact) subsets of X. Let $T : X \to 2^X$ be a multi-mapping. Given $U \in \mathcal{U}$, let

$$U_T = \{x \in X : x \in U[T(x)]\}$$

and
$$U'_T = \{x \in X : (x, y) \in U \text{ for all } y \in T(x)\}.$$

The families $\{U_T \times U_T \cup \Delta : U \in \mathcal{U}\}$ and $\{U'_T \times U'_T \cup \Delta : U \in \mathcal{U}\}$ constitute bases for uniformities on X. (Here Δ denotes the diagonal of $X \times X$). We denote these uniformities by $\mathcal{U}_T$, $\mathcal{U}'_T$ respectively. Given $U \in \mathcal{U}$, let

$$\dot{U} = \{(A, B) \in 2^X \times 2^X : A \times B \subset U\} \cup \Delta$$

and
$$\tilde{U} = \{(A, B) \in 2^X \times 2^X : A \subset U[B] \text{ and } B \subset U[A]\}.$$

The families

$$\{\dot{U} : U \in \mathcal{U}\}, \{\tilde{U} : U \in \mathcal{U}\}$$

constitute bases for uniformities on 2^X. We denote these uniformities by $\hat{\mathcal{U}}$, $\tilde{\mathcal{U}}$ respectively. Let $\mathcal{D} = \{d_i : i \in I\}$ be a family of uniformly continuous pseudometrics on X such that the family $\{B(i, \epsilon) : i \in I, \epsilon > 0\}$, where

$$B(i, \epsilon) = \{(x, y) : d_i(x, y) < \epsilon\}$$

is a base for $\mathcal{U}$. Such a family $\mathcal{D}$ is called an augmented associated family for the uniformity $\mathcal{U}$. It is well known that for each uniformity on X, there exists an augmented associated family $\mathcal{D}$ (cf. Thron 1966, p. 177). For each $i \in I$ and $A, B \in 2^X$, let

$$\delta_i(A, B) = \sup \{d_i(a, b) : a \in A, b \in B\}$$

and $$D_i(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d_i(a, b), \sup_{b \in B} \inf_{a \in A} d_i(a, b) \right\}.$$

Let $$\hat{B}(i, \epsilon) = \{(A, B) \in 2^X \times 2^X : \delta_i(A, B) < \epsilon\} \cup \Delta$$

$$\tilde{B}(i, \epsilon) = \{(A, B) \in 2^X \times 2^X : D_i(A, B) < \epsilon\}$$

and let

$$\hat{\mathcal{B}} = \{\hat{B}(i, \epsilon) : i \in I, \epsilon > 0\}$$

$$\tilde{\mathcal{B}} = \{\tilde{B}(i, \epsilon) : i \in I, \epsilon > 0\}.$$

It is easily verified that the families $\hat{\mathcal{B}}$, $\tilde{\mathcal{B}}$ constitute bases for uniformities on 2^X that are uniformly equivalent to the uniformities $\hat{\mathcal{U}}$, $\tilde{\mathcal{U}}$ respectively.

Theorem 2.1 — Let $(X, \mathcal{U})$ be a nonempty complete uniform space. Suppose that $T : (X, \mathcal{U}_T) \to (2^X, \hat{\mathcal{U}})$ is uniformly continuous and satisfies (i) $x \neq y$ and $T(x)$, $T(y)$ are not singleton sets imply $T(x) \neq T(y)$; (ii) U_T is a nonempty closed subset of X for each $U \in \mathcal{B}$.

Then T has a fixed point. Furthermore, if X is Hausdorff, then the fixed point is unique.

PROOF : We consider the family $\mathcal{F} = \{U_T : U \in \mathcal{B}\}$. By hypothesis, $\mathcal{F}$ is a family of nonempty closed sets with finite intersection property. We assert that $\mathcal{F}$ contains small sets. Let $W \in \mathcal{U}$. Pick up $U \in \mathcal{B}$ such that $U \cdot U \cdot U \subset W$. Since $T : (X, \mathcal{U}_T) \to (2^X, \hat{\mathcal{U}})$ is uniformly continuous, there is a $V \in \mathcal{B}$ such that $(u, v) \in V_T \times V_T \cup \Delta$ implies $(T(u), T(v)) \in \hat{U}$. Let $H = U \cap V$. Then $H_T \in \mathcal{F}$. Let $x, y \in H_T$. Then $(x, z_x) \in H$, $(y, z_y) \in H$ for some $z_x \in T(x)$ and $z_y \in T(y)$.

Since $(x, y) \in V_T \times V_T$, $(T(x), T(y)) \in \hat{U}$ and hence $(z_x, z_y) \in U$. Thus $(x, y) \in H \cdot U \cdot H \subset W$. Therefore, $H_T \times H_T \subset W$ and $\mathcal{F}$ contains small sets. Since $(X, \mathcal{U})$ is complete $\cap \mathcal{F} \neq \phi$. Let $x \in \cap \mathcal{F}$. Then

$$x \in \cap \{U[T(x)] : U \in \mathcal{B}\} = \overline{T(x)} = T(x)$$

and x is a fixed point of T. Now suppose that X is Hausdorff and let u, v be fixed points of T. Given $V \in \mathcal{B}$, there is a $U \in \mathcal{B}$ such that $(T(x), T(y)) \in \hat{V}$ whenever $(x, y) \in U_T \times U_T \cup \Delta$. Since $(u, v) \in U_T \times U_T$, one has $(u, v) \in V$ for each $V \in \mathcal{B}$.

Therefore, $(u, v) \in \cap \mathcal{B} = \Delta$. Hence $u = v$, and the proof is complete.

Theorem 2.1 generalizes a point-valued result of Wong (1973, p. 97) to multi-mappings.

It is easily seen that $T : (X, \mathcal{U}_T) \to (2^X, \hat{\mathcal{U}})$ is uniformly continuous if and only if for each $i \in I$ and $\epsilon > 0$, there is a $j \in I$ and $\delta(\epsilon) > 0$ such that $x, y \in X$ and

$$d_j(x, T(x)) + d_j(y, T(y)) < \delta(\epsilon)$$

imply either

$$T(x) = T(y) \quad \text{or} \quad \delta_i(T(x), T(y)) < \epsilon.$$

(Here $d_j(x, T(x)) = \inf \{d_j(x, y) : y \in T(x)\}$.)

We denote the class of multi-mappings $T : X \to Cpt(X)$ satisfying the above criterion by $\hat{\mathcal{D}}(X)$. An easy application of Theorem 2.1 yields:

Theorem 2.2 — Let X be a nonempty complete uniform space. Suppose $T \in \hat{\mathcal{D}}(X)$ and satisfies:

(i) $x \neq y$ and $T(x)$, $T(y)$ are not singleton sets imply $T(x) \neq T(y)$;

(ii) the set $\{x \in X : d_i(x, T(x)) \leqslant \epsilon\}$ is nonempty and closed for each $i \in I$ and $\epsilon > 0$.

Then T has a fixed point which is unique if X is Hausdorff.

Corollary 2.3 — Let $T : X \to Cpt(X)$ and suppose that

(i) for each $i \in I$, there is a constant k_i, $0 \leqslant k_i < 1$ such that given $x, y \in X$, we have $D_i(T(x), T(y)) \leqslant k_i d_i(x, y)$;

(ii) for each $i \in I$, there exist a $j \in I$ and some $\alpha_i > 0$ such that

$$\delta_i(T(x), T(y)) \leqslant \alpha_i \{d_j(x, T(x)) + d_j(y, T(y))\}$$

for $x, y \in X$ with $T(x) \neq T(y)$;

(iii) $x \neq y$ and $T(x)$, $T(y)$ are not singleton sets imply $T(x) \neq T(y)$.

Then T has a unique fixed point.

PROOF : By hypothesis (ii), $T \in \hat{\mathscr{D}}(X)$. By using a technique due to Nadler (1969, p. 479), one obtains a sequence $\{x_n\}$ satisfying $x_n \in T(x_{n-1})$ and

$$d_i(x_n, x_{n+1}) \leqslant k_i^n \, d_i(x_0, x_n) \text{ for } n = 1, 2, 3, \ldots .$$

Thus, the set $\{x \in X : d_i(x, T(x)) \leqslant \epsilon\}$ is nonempty for each $i \in I$ and $\epsilon > 0$; for each $i \in I$, $x \to d_i(x, T(x))$ is continuous. This follows from the inequalities

$$| d_i(x, T(x)) - d_i(y, T(y)) | \leqslant d_i(x, y) + | d_i(y, T(x)) - d_i(y, T(y)) |$$

and

$$| d_i(y, T(x)) - d_i(y, T(y)) | \leqslant D_i(T(x), T(y)).$$

Hence $\{x \in X : d_i(x, T(x)) \leqslant \epsilon\}$ is closed for each $i \in I$ and $\epsilon > 0$. Therefore, by Theorem 2.2, there exists $x \in X$ such that $x \in T(x)$. Suppose there exist two points x, y with $x \neq y$ and $x \in T(x)$, $y \in T(y)$. Then $\delta_i(T(x), T(y)) = 0$ for all $i \in I$ and this implies $x = T(x) = T(y) = y$.

Remark : In case of single-valued mapping, condition (*ii*) of the above corollary follows from condition (*i*) and hence Corollary 2.3 is a multi-valued extension of Tarafdar's theorem (1974, Theorem 1.1).

Example — Let $X = \{x \in \mathbb{R} : x \geqslant 0\}$. Then X is a complete metric space with the usual metric.

Define $T : X \to Cpt(X)$ by $T(x) = \left[0, \frac{x}{2}\right]$ for all $x \in X$. Then

$$D(T(x), T(y)) = \tfrac{1}{2}(y - x) = \tfrac{1}{2} d(x, y)$$

and

$$\delta(T(x), T(y)) = \frac{y}{2} \leqslant \frac{x}{2} + \frac{y}{2} = d(x, T(x)) + d(y, T(y)).$$

Now X satisfies all the conditions of Corollary 2.3 and the unique fixed point is $x = 0$.

Theorem 2.4 — Let $(X, \mathcal{U})$ be a nonempty complete uniform space. Suppose $T : (X, \mathcal{U}'_T) \to (2^X, \hat{\mathcal{U}})$ is uniformly continuous and that U'_T is a nonempty closed subset of X for each $U \in \mathscr{B}$. Then T has a fixed point.

The proof of Theorem 2.4 is similar to that of Theorem 2.1 and hence is omitted.

Remark : If X is Hausdorff, then $\cap \{U'_T : U \in \mathscr{B}\}$ contains a unique point u such that $T(u) = u$. However, in general, the uniqueness of fixed points is not ensured.

It is easily seen that $T : (X, \mathcal{U}_T') \to (2^X, \tilde{\mathcal{U}})$ is uniformly continuous if and only if for each $i \in I$ and $\epsilon > 0$, there is a $j \in I$ and $\delta(\epsilon) > 0$ such that $x, y \in X$ and $\delta_j(x, T(x)) + \delta_j(y, T(y)) < \delta(\epsilon) \Rightarrow D_i(T(x), T(y)) < \epsilon$. (Here $\delta_j(x, T(x)) = \sup \{d_j(x, y) : y \in T(x)\}$.) We denote the class of multi-mappings $T : X \to 2^X$ satisfying the above criterion by $\tilde{\mathcal{D}}'(X)$. From Theorem 2.4, we easily obtain:

Theorem 2.5 — Let X be a nonempty complete uniform space. Suppose $T \in \tilde{\mathcal{D}}'(X)$ and satisfies:

The set $\{x \in X : \delta_i(x, T(x)) \leqslant \epsilon\}$ is nonempty and closed for each $i \in I$ and $\epsilon > 0$. Then T has a fixed point.

Corollary 2.6 — Let $T : X \to 2^X$ and suppose that for each $i \in I$, there are constants $k_i, \alpha_i \geqslant 0$, $k_i < 1$ such that:

(i) $\delta_i(T(x), T(y)) \leqslant k_i\, \delta_i(y, T(x))$ for $x, y \in X$, and

(i′) $\delta_i(T(x), T(y)) \leqslant \alpha_i d_i(x, y)$ for $x, y \in X$ with $T(x) \neq T(y)$;

(ii) for each $i \in I$, there is a $j \in I$ and some $\beta_j \geqslant 0$ such that

$D_i(T(x), T(y)) \leqslant \beta_j\{\delta_j(x, T(x)) + \delta_j(y, T(y))\}$ for $x, y \in X$. Then T has a fixed point.

PROOF : By hypothesis (*ii*), $T \in \tilde{\mathcal{D}}'(X)$. Let $x_o \in X$. Define $x_n \in T(x_{n-1})$ for $n = 0, 1, 2, \ldots$. Then $\delta_i(x_n, T(x_n))$ tends to 0. This follows from (*i*).

Hence $\{x \in X : \delta_i(x, T(x)) \leqslant \epsilon\}$ is nonempty for each $i \in I$ and $\epsilon > 0$. The continuity of mapping $x \to \delta_i(x, T(x))$ for each $i \in I$ follows easily from (*i′*).

We recall that a mapping $T: X \to 2^X$ is said to be 'upper semi-continuous' (u.s.c.) if $T^{-1}(K) = \{x \in K : T(x) \cap K \neq \phi)$ is a closed set for each closed subset K of X. It is easily verified that if X is a compact uniform space, then T is u.s.c. if and only if for each net $x_\lambda \to x_0$ and a net $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y_0$, one has $y_0 \in T(x_0)$. The following result is essentially known (cf. Ky Fan 1959, p. 128). We recall it here in the present setting for the sake of completeness.

Theorem 2.7 — Let $(X, \mathcal{U})$ be a nonempty compact uniform space. Suppose that T satisfies:

(i) $T : X \to 2^X$ is u.s.c.;

(ii) for each $U \in \mathcal{B}$, $U_T \neq \phi$.

Then T has a fixed point.

PROOF : We consider the family $\mathcal{F} = \{\bar{U}_T : \mathrm{U} \in \mathcal{B}\}$ of non-empty closed sets. Evidently, it has finite intersection property. Since X is compact, one has $\cap\, \mathcal{F} \neq \phi$.

Let $x \in \cap \mathcal{F}$. We partially order the family $\{U(x) : U \in \mathcal{B}\}$ of neighbourhoods of x by the reversed set inclusion. Pick up $W \in \mathcal{B}$ such that $W(x)$ is in this family. Let $U \in \mathcal{B}$ be such that $U . U \subset W$. Since $x \in \bar{U}_T$, there is $y_W \in U_T$ and $z_W \in Ty_W$ such that $(x, y_W) \in U$, $(y_W, z_W) \in U$ and consequently $(x, z_W) \in W$. The nets y_W and z_W converge to x. Since $z_W \in Ty_W$ and T is u.s.c., we have $x \in T(x)$.

In case X is a compact uniform space, we note that if $T : (X, \mathcal{U}) \rightarrow (2^X, \tilde{\mathcal{U}})$ is uniformly continuous, then it is u.s.c. This follows from the remark preceding Theorem 2.7. Again, it is easily observed that $T : (X, \mathcal{U}) \rightarrow (2^X, \tilde{\mathcal{U}})$ is uniformly continuous if and only if for each $i \in I$ and $\epsilon > 0$, there is $j \in I$ and $\delta(\epsilon) > 0$ such that $D_i(T(x), T(y)) < \epsilon$ whenever $d_j(x, y) < \delta(\epsilon)$. Denote the class of multi-mappings $T : X \rightarrow 2^X$ satisfying this condition by $\tilde{\mathcal{D}}(X)$.

Corollary 2.8 — Let X be a nonempty compact uniform space. Suppose $T \in \tilde{\mathcal{D}}(X)$. and satisfies:

The set $\{x \in X : d_i(x, T(x)) < \epsilon\}$ is nonempty for each $i \in I$ and $\epsilon > 0$. Then T has a fixed point.

Corollary 2.9 — Let X be a nonempty compact uniform space. Let $T : X \rightarrow 2^X$ and suppose that for each $i \in I$, there is a constant k_i, $0 \leqslant k_i < 1$ such that $D_i(T(x), T(y)) \leqslant k_i d_i(x, y)$ for all $x, y \in X$. Then T has a fixed point.

PROOF : Evidently, $T \in \tilde{\mathcal{D}}(X)$. By using a technique due to Nadler (1969, p. 479), one obtains a sequence $\{x_n\}$ satisfying

$$x_n \in T(x_{n-1}) \text{ and } d_i(x_n, x_{n+1}) \leqslant k_i^n d_i(x_0, x_1)$$

for $n = 1, 2, 3, \ldots$. Thus the set $\{x \in X : d_i(x, T(x)) < \epsilon\}$ is nonempty for each $i \in I$ and $\epsilon > 0$, and this completes the proof.

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