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OPERATORS HAVING FINITE SPECTRUM

J. M. Patel

Department of Mathematics, University School of Sciences, Gujarat University, Ahmedabad 380009

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This paper is devoted to the study of an operator whose spectrum is finite under certain conditions of similarity.

1. Introduction

The study of operators similar to their adjoints has been followed by several authors. Williams (1969) gave an elegant and simple proof of the following theorem.

Theorem A — If T is any operator such that $S^{-1}TS = T^*$ where $0 \notin \overline{W(S)}$, then the spectrum of T is real.

Using the same technique, Patel (1974) generalized Theorem A as under:

Theorem B — If T is any operator such that $T^{*q} = S^{-1}T^pS$ where $0 \notin \overline{W(S)}$ and p, q are integers then $\lambda^p = \lambda^{*q}$ for $\lambda \in \sigma(T)$.

In this paper, we discuss various implications of Theorem B and some related results.

2. Notations and Terminology

We shall consider a (bounded linear) operator T on a Hilbert space H. For an operator T, we denote the closure of the numerical range by $\overline{W(T)}$, the point spectrum by $\sigma_p(T)$, the approximate point spectrum by $\sigma_a(T)$, the spectrum by $\sigma(T)$, the Weyl spectrum by $\sigma_w(T)$, the set of isolated point of $\sigma(T)$ with finite multiplicity by $\sigma_{0f}(T)$, the convex hull of $\sigma(T)$ by con $\sigma(T)$, the spectral radius by $r(T)$, the numerical radius by $w(T)$, the null space by $N(T)$, the complex conjugate of a complex number z by z^* and the complex plane by C.

An operator T is said to be normal if $T^*T = TT^*$, unitary if $T^*T = TT^* = I$, hyponormal if $T^*T \geqslant TT^*$, paranormal if $\| Tx \|^2 \leqslant \| T^2x \| \, \| x \|$ for $x \in H$, normaloid if $\| T \| = r(T)$, convexoid if $\overline{W(T)} = \text{con}\,\sigma(T)$, spectraloid if $r(T) = w(T)$. An operator T is said to satisfy the growth condition (G_1) if $(T - zI)^{-1}$ is normaloid for each $z \notin \sigma(T)$. A unitary operator U is called cramped if $\sigma(U)$ is contained in an arc of the

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unit circle of length less then π. A set S is called a spectral set for T, if $\sigma(T) \subset S$ and $\| f(T) \| \leqslant \sup \{ | f(z) | : z \in S\}$ for every rational function f with poles off S.

Throughout the paper, we shall say that T satisfies the condition (J_{pq}) with respect to an operator S if $S^{-1}T^pS = T^{*q}$, where $0 \notin \overline{W(S)}$ and p, q are integers. If T is invertible, then we shall denote a polynomial of the form $\sum_{i=1}^{m} a_i z^i + \sum_{j=1}^{n} b_j z^{-j}$ on $\sigma(T)$ by $f(z, z^{-1})$ and an operator $\sum_{i=1}^{m} a_i T^i + \sum_{j=1}^{n} b_j T^{-j}$ by $f(T, T^{-1})$ where a_i, b_j are scalars.

3. Theorems and Their Related Results

The following theorem is a generalization of Theorem B:

Theorem 1 — If T is an invertible operator such that

$$f_1(T, T^{-1})\, S = S\, f_2(T, T^{-1})^*, \quad \text{for} \quad 0 \notin \overline{W(S)}$$

then

$$f_1(z, z^{-1}) = f_2(z, z^{-1})^*, \quad z \in \sigma(T).$$

Proof: Since $\sigma(T) = \sigma_a(T) \cup \sigma_a(T^*)^*$, it is sufficient to prove the theorem for $z \in \sigma_a(T)$. Let $\{x_n\}$ be a sequence of unit vectors such that $\| (T - zI)\, x_n \| \to 0$. Then

$$\| [f_i(T, T^{-1}) - f_i\,(z, z^{-1})\, I]\, x_n \| \to 0 \ \text{ for } \ i = 1, 2.$$

After some computational work, we get

$$| [f_1(z, z^{-1}) - f_2(z, z^{-1})^*]\, (S^{-1}x_n, x_n) \,| \to 0 \ \text{ as } \ n \to \infty.$$

As $0 \notin \overline{W(S^{-1})}$, the result follows.

Remark : Under the condition of theorem, we have

$$\sigma(T) \subset \{z \in C : f_1(z, z^{-1}) = f_2(z, z^{-1})^*\}.$$

Taking $f_1(T, T^{-1}) = T^p$ and $f_2(T, T^{-1}) = T^q$, we have the following:

Corollary 1 — If T satisfies the condition (J_{pq}) w.r.t. S, then

(a) $\sigma(T)$ is a finite subset of $\{0\} \cup \{z \in C : z^{p+q} = 1\}$ where p and q are unequal positive integers.

(b) $\sigma(T) \subset \{z \in C : z^{p+q} = 1\}$ where $| p | \neq | q |$ and T is invertible.

(c) $\sigma(T^p)$ is real where $p = q$ and $\sigma(T) \subset$ unit circle whose $p + q = 0$.

It is interesting to see that the bilateral shift operator satisfies the condition $(J_{-1,1})$ w.r.t. I and its spectrum is the unit circle and for an operator T on

$$H = H_1 \oplus H_1 \oplus H_1 \oplus \ldots,$$

where $T = A \oplus A \oplus A \oplus \ldots, A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ on two dimensional unitary space H_1, T satisfies the condition $(J_{-1,1})$ w.r.t. I and $\sigma(T) = \{-i, i\}$ which is a finite set. That is, $\sigma(T)$ is not necessarily finite even if $p \pm q = 0$.

The next corollary follows from the result of Saitô (1972, Lemma 3.1, p. 574).

Corollary 2 — For a convexoid operator T, if $W(T)$ is closed and T satisfies the condition (J_{pq}) w.r.t. S where $| p | \neq | q |$, then

(1) $N(T - zI) = N(T^* - z^*I)$ and

(2) ascent of $(T - zI)$ is 1 for $z \in \sigma(T) - \{0\}$.

Using the result of Berberian (1970, Lemmas 2 and 3), we have the following:

Corollary 3 — If T is a restriction-convexiod satisfying the condition (J_{pq}) w.r.t. S where $| p | \neq | q |$, then each point of $\sigma(T)$ is an eigenvalue and T is normal.

Remark: The above corollary is also applicable for restriction-transloid operators and in particular for hyponormal operators.

Using the result appeared in (Stampfli 1965, Theorem C), we have the following:

Corollary 4 — If T satisfies the conditions (G_1) and (J_{pq}) w.r.t. S where $| p | \neq | q |$ then T is normal.

Since $\sigma(T)$ lies on the unit circle under the conditions of Corollary 4, we have

Corollary 5 — If T is invertible and satisfies the hypothesis of Corollary 4, then T is unitary.

This corollary can be proved alternatively with the help of the result of Embry (1970).

In Corollary 4, we put the restriction $| p | \neq | q |$. Using Andô's result (1972) we have the following result for paranormal operators.

Corollary 6 — If T is paranormal and $T^p = T^{*q}$ where p and q are integers then T is normal.

Using the result of Luecke (1972, Corollary 2) we get the following:

Corollary 7 — If T satisfies the condition (J_{pq}) w.r.t. S where

$$| p | \neq | q | \text{ and } \| (T - zI)^{-1} \| = \frac{1}{d(z, W(T))}$$

for $z \notin \overline{W(T)}$ then T is a scalar multiple of identity.

The next corollary follows from Saitô (1972, Corollary 2.1, p. 555).

Corollary 8 — If T is a contraction operator satisfying the condition (J_{pq}) w.r.t. S where $p > 1$ or $p \leqslant -1$ then each non-zero $z \in \sigma(T)$ is normal approximate eigenvalue.

Corollary 9 — If T satisfies the condition (J_{pq}) w.r.t. S, where $|p| \neq |q|$, T is restriction-convexoid and each eigenspace of T is infinite-dimensional then $\sigma(T) = \sigma_w(T)$.

This result follows from the fact that $\sigma(T) - \sigma_w(T) \subset \sigma_{0f}(T)$ and Corollary 3.

Corollary 10 — If $T = A + iB$ is a Cartesian-decomposition of T, $0 \notin \overline{W(A)}$ and AB is convexoid, then T is normal.

PROOF: Since $(AB)^* = BA = A^{-1}(AB)A$, $\sigma(AB)$ is real by Corollary 1. The convexoidity of AB implies that AB is self-adjoint. This is the sufficient condition for normality.

We mention below one interesting result regarding nilpotent operator.

Theorem 2 — If T is a nilpotent operator such that $T^{p+1} = 0$ then T does not satisfy the condition $(J_{p,1})$.

PROOF: Suppose that there is an operator S such that $S^{-1}T^pS = T^*$ and $0 \notin \overline{W(S)}$. Therefore

$$0 = S^{-1}T^{p+1}S = (S^{-1}T^pS)(S^{-1}TS) = T^*(S^{-1}TS).$$

Hence $T^*S^{-1}T = 0$ and for all $x \in H$, $(S^{-1}Tx, Tx) = 0$ a contradiction.

The operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ defined on two dimensional unitary space being nilpotent, does not satisfy the condition $(J_{1,1})$. Incidently it is non-convexoid and binormal (i.e. T^*T commutes with TT^*). Here one can investigate the various classes of operators for which the condition (J_{pq}) is satisfied w.r.t. some operator S. In this direction, the following example shows that there exists a non-normaloid, non-convexoid and non-spectraloid operator T such that the condition (J_{pq}) w.r.t. some operator S holds for T.

Example — Consider $T = \frac{1}{2}\begin{pmatrix} -1 & -3 \\ 1 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ on two dimensional unitary space H, we have $ST^2S^{-1} = T^*$ and $0 \notin \overline{W(S)}$. Moreover $\sigma(T) = \{w, w^2\}$, where $w^3 = 1$, $r(T) = 1$, $\| T \| = 5/2$ and $w(T) = 19/12$.

Theorem 3 — If T is an invertible contraction operator satisfying the condition (J_{pq}) w.r.t. S where $p \neq q$ and $\sigma(T)$ is a spectral set for T then T is unitary.

PROOF : $\sigma(T)$ being a spectral set and lying on the unit circle, we have

$$1 = \| TT^{-1} \| \leqslant \| T^{-1} \| \leqslant \text{Sup} \{ | z^{-1} | : z \in \sigma(T)\} = 1.$$

Now $\| x \| = \| T^{-1}(Tx) \| \leqslant \| Tx \| \leqslant \| x \|$, since $\| T^{-1} \| = 1$. This implies that $\| Tx \| = \| x \|$. Together with this and invertibility of T, T becomes unitary.

The following theorem is a natural generalization of the result of De Prima (1974).

Theorem 4 — If T and T^{-1} both are spectraloid and T satisfies the condition (J_{pq}) w.r.t. S, where $p \neq q$, then T is unitary.

PROOF : By Corollary 1, $\sigma(T)$ and $\sigma(T^{-1})$ both lie on the unit circle and by the hypothesis, $1 = r(T) = w(T)$ and $1 = r(T^{-1}) = w(T^{-1})$. Now T becomes unitary by the result of Stampfli (1967, Corollary 1).

An alternative proof of Corollary 1 can be given if S is self-adjoint under the weaker condition as follows:

Theorem 5 — If T satisfies the condition $S^{-1}T^pS = T^{*q}$ for $| p | \neq | q |$ where S is an invertible self-adjoint operator, then $\sigma(T)$ lies on the unit circle.

PROOF : It is sufficient to prove the theorem for positive integers p and q. Since $S^{-1}T^pS = T^{*q}$ and $ST^{*p}S^{-1} = T^q$, we have $ST^{*p^2}S^{-1} = T^{pq}$ and

$$ST^{*q^2}S^{-1} = T^{pq}, \quad \text{i.e. } T^{*p^2} = T^{*q^2} \text{ or } T^{p^2-q^2} = I.$$

By spectral mapping theorem, $z^{p^2-q^2} = 1$ for each $z \in \sigma(T)$. This gives us that $\sigma(T)$ is a finite set and lies on the unit circle.

Using the technique of Singh and Kanta (1973, Theorem 1), we give below an alternative proof of a Theorem of Patel (1974).

Theorem 6 — If T is an invertible operator satisfying the condition (J_{pq}) w.r.t. S where $p \neq q$ then T is similar to unitary.

PROOF : Since $0 \in \overline{W(S)}$, we can assume that Re $\overline{W(T)} > \epsilon$ for some positive real number ϵ. Hence $P = S + S^*$ is positive and invertible.

Defining $J(S) = T^{p+q}S - ST^{*p+q}$, we have $J(S)^* = J(S^*)$. This gives us that $T^{p+q}S^* = S^*T^{*p+q}$.

Taking adjoints, we have $ST^{*p+q} = T^{p+q}S$. Hence

$$T^{p+q}P = T^{p+q}(S + S^*) = (S^* + S)\, T^{*p+q} = PT^{*p+q}.$$

Since $0 \notin \overline{W(P)}$, T^{p+q} is similar to a self-adjoint operator $P^{-1/2}T^{p+q}P^{1/2}$.

But by Corollary 1, we have $z^{p+q} = 1$ for $z \in \sigma(T)$. Hence

$$\sigma(T^{p+q}) = \sigma(P^{-1/2}T^{p+q}P^{1/2}) = \{1\} \text{ and } P^{-1/2}T^{p+q}P^{1/2} = I.$$

This gives us that $T^{p+q} = P^{1/2}P^{-1/2} = I$. Using the result of Kurepa (1962), T is similar to unitary.

For $p = -1$ and $q = 1$, T is similar to unitary by the result of Singh and Kanta (1973).

We refine the result of Patel (1973, Theorem 6) as follows:

Theorem 7 — Let T be a left invertible operator with left inverse T_1. If there is a self-adjoint operator S such that $T^* = S^{-1}T_1^p S$ where $0 \notin \overline{W(S)}$ and $p > 1$, then T is similar to unitary.

PROOF : Here $T_1T = I$. Using the theorem of Patel (1973, Theorem 6), we have T similar to isometry say A and $\sigma(T)$ is finite for $p > 1$. Since A is isometry with $\sigma(A)$ finite, A is normal. Hence A is unitary, the desired conclusion.

Remark : Using Theorem 6, we can generalize Theorem 7 when $T^{*q} = S^{-1}T_1^p S$ where $0 \notin \overline{W(S)}$ and $p \neq q$.

Theorem 8 — Let T be an invertible operator. If T is similar to a unitary operator U such that $U^{p+q} = I$ where $p \neq q$ then the condition (J_{pq}) for T is satisfied w.r.t. some S with $0 \notin \overline{W(S)}$.

PROOF : Suppose $U = A^{-1}TA$ for some invertible operator A. Now

$$I = U^pU^q = (A^{-1}T^pA)\,(A^*T^{*-q}A^{*-1})$$

gives us that $AA^*T^{*q} = T^pAA^*$.

For $S = AA^*$, we have $T^{*q} = S^{-1}T^pS$ and $0 \notin \overline{W(S)}$.

Now we prove the following theorem in which T satisfies the condition (J_{pp}) where $p = q$.

Theorem 9 — If a paranormal operator T satisfies the condition (J_{pq}) w.r.t. a cramped unitary operator U for some non-zero integer p then T turns out to be normal.

PROOF : Here $U^{-1}\,T^pU = T^{*p}$ and $0 \notin \overline{W(U)}$. Using the result of Berberian (1962), we have T^p is self-adjoint. Now T being para-normal, T becomes normal by Andô (1972).

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