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QUASINORMAL COMPOSITION OPERATORS ON l^2_p

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The composition operator C_T on a weighted sequence space l^2_p induced by the mapping T from the set N of natural numbers into itself is a bounded linear transformation defined by $C_T f = f \circ T$. This paper is a report on normal, quasinormal, isometry and unitary composition operators. It turns out that these four classes of composition operators coincide on suitable weighted sequence spaces.

1. PRELIMINARIES

Let N denote the set of non-zero positive integers and let $\{p_n : n \in N\}$ be a sequence of strictly positive numbers such that $\Sigma\, p_n < \infty$. Then we can define measure λ on the σ-algebra of all subsets of N by $\lambda(E) = \sum_{n \in E} p_n$ for every $E \subseteq N$.

Let l^2_p be defined as

$$l^2_p = \{g \mid g : N \to C \text{ and } \sum_{n=1}^{\infty} p_n \mid g(n) \mid^2 = \int_N \mid g \mid^2 d\lambda < \infty\}.$$

Then l^2_p is a Hilbert space under pointwise addition and scalar multiplication with inner product defined by

$$\langle f, g \rangle = \sum_{n=1}^{\infty} p_n f(n) \overline{g(n)} \text{ for all } f, g \in l^2_p.$$

If T is a mapping from N into itself, we define the composition transformation C_T on l^2_p into the space of all complex valued function on N by

$$C_T f = f \circ T.$$

If the range of C_T is in l^2_p and C_T is bounded, then we call C_T a composition operator induced by T. By $B(l^2_p)$, we denote the Banach algebra of all bounded linear operators on l^2_p.

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In Theorem 1 of section 2 of this note we show that a composition operator is quasinormal if and only if it is an isometry, and in Theorem 2 we prove that a composition operator is an isometry if and only if it is unitary. The two theorems lead to the conclusion that in $B(l_p^2)$ the sets of normal, quasinormal, isometric and unitary composition operators coincide.

2. Quasinormal Composition Operators

Definition — An operator $A \in B(H)$ is said to be quasinormal if $AA^*A = A^*AA$, where A^* is the adjoint of A.

In $B(l^2)$ the set of quasinormal composition operators is a bigger set compared to the set of quasinormal composition operators in $B(l_p^2)$. In the first case it includes the set of all invertible composition operators (Singh and Komal 1978), while in the latter it equals the set of isometric composition operator. This is proved in the following theorem.

Theorem 1 — Let $C_T \in B(l_p^2)$. Then C_T is quasinormal if and only if C_T is an isometry.

Proof: The sufficiency is obvious. We prove the necessary part. Let C_T be quasinormal. Then it commutes with M_{f_0} by Theorem 3 of Singh (1975a), where M_{f_0} is the multiplication operator induced by f_0, the Radon Nikodym derivative of the measure λT^{-1} with respect to the measure λ. Let $X_{\{n_1\}}$ be the characteristic function of the singleton set $\{n_1\}$. Then

$$M_{f_0} C_T X_{\{n_1\}} = C_T M_{f_0} X_{\{n_1\}}$$

or $$M_{f_0} X_{T^{-1}(\{n_1\})} = C_T(f_0 X_{\{n_1\}})$$

or $$f_0 X_{T^{-1}(\{n_1\})} = f_0(n_1)\, X_{T^{-1}(\{n_1\})}.$$

This implies that $f_0(m) = f_0(n_1)$ for all $m \in T^{-1}(\{n_1\})$ and for all $n_1 \in N$. A little computation shows that $f_0(m) = f_0(n_1)$ for all $m \in \bigcup_{i=1}^{\infty} T^{-i}(\{n_1\})$, where

$$T^{-i}(\{n_1\}) = T^{-1}\{T^{-(i-1)}(\{n_1\})\}.$$

Let $F_0^1 = \bigcup_{i=1}^{\infty} T^{-i}(\{n_1\})$ and $F_p^1 = \bigcup_{i=1}^{\infty} T^{-i}(T^p(n_1))$, where $T^p(n_1) = T \circ T^{p-1}(n_1)$. Then $T^{-1}(E_1) = E_1$, where $E_1 = \bigcup_{p=0}^{\infty} F_p^1$. Since $F_n^1 \subseteq F_{n+1}^1$ for all $n_1 \in N \cup \{0\}$, it follows that $f_0(m) = f_0(n_1)$ for all $m \in E_1$. By Radon Nikodym theorem

$$\lambda T^{-1}(E_1) = \int_{E_1} f_0 d\lambda,$$

which gives

$$\lambda(E_1) = f_0(n_1) \,.\, \lambda(E_1).$$

Since λ is a finite measure, $f_0(n_1) = 1$. Hence $f_0(m) = 1$ for all $m \in E_1$. If $E_1 = N$, the proof is complete. In case $\lambda(N \setminus E_1) \neq 0$, then let $n_2 \in N \setminus E_1$. Thus as before $f_0(m) = 1$ for all $m \in E_2$ where E_2 is constructed for n_2 in the same fashion as E_1 is constructed for n_1. Again if $\lambda(N \setminus (E_1 \cup E_2)) \neq 0$, we repeat the process finitely or infinitely many times till $\lambda(N \setminus \cup E_i) = 0$. This implies that $f_0(m) = 1$ for all $m \in \cup E_i = N$. Hence by Theorem 1 of Singh (1975b) C_T is an isometry. This completes the proof of the theorem.

In the next theorem we show the equivalence of isometric and unitary composition operators.

Theorem 2 — Let $C_T \in B(l^2_p)$. Then C_T is an isometry if and only if C_T is unitary.

PROOF : The sufficiency is again trivial. We prove the necessary part. Since $\Sigma p_i < \infty$, without loss of generality we can suppose that $p_1 \geqslant p_2 \geqslant p_3 \ldots$. Let $A_i = \{n : \lambda(\{n\}) = p_i\}$. It can be easily shown that collection $\mathcal{F}$ of all distinct A_i's is a partition of N. If $A_i \in \mathcal{F}$, then it contains finitely many natural numbers since $\Sigma p_i < \infty$. Suppose C_T is an isometry. Then $f_0(n) = 1$ or $\lambda T^{-1}(\{n\}) = \lambda(\{n\})$ for every $n \in N$. Hence it follows that the restriction of T to A_i is a bijection from A_i onto A_i. Since $\cup A_i = N$, we can conclude that T is a bijection on N. Hence by a result of Singh (1975b) C_T has dense range, and therefore it is unitary. This completes the proof.

In the case of l^2 every quasinormal composition operator is not normal as is clear from the following example.

Example — Let $T : N \to N$ be defined by

$$T(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd,} \\[2ex] \dfrac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Then C_T is quasinormal on l^2, but, since T is not invertible, C_T is not normal [see Corollary 4.1 of Singh and Kumar (1978)].

It is interesting to note in the light of Theorem 1 and Theorem 2 that every quasinormal composition operator on l^2_p is normal. Thus we can state the following result.

Theorem 3 — Let $C_T \in B(l^2_p)$. Then the following are equivalent :

(i) C_T is normal,

(ii) C_T is quasinormal,

(iii) C_T is an isometry,

(iv) C_T is unitary.

Corollary — Let $C_T \in B(l^2_a)$. Then the normal, quasinormal, isometric and unitary composition operators are all equal to the identity operator.

PROOF : The proof follows from the above theorem and Theorem 2.2 of Singh and Gupta (1978).

To illustrate the theory we quote an example.

Example — Let $T : N \to N$ be defined as

$$T(n) = \begin{cases} n + 1, & \text{if } n \text{ is odd,} \\ n - 1, & \text{if } n \text{ is even.} \end{cases}$$

Let the sequence $\{p_n\}$ of weights be given by $(1/2^2, 1/2^2, 1/3^2, 1/3^2, \ldots)$. Then

$$f_0(n) = \lambda T^{-1}(\{n\})/\lambda(\{n\}) = 1$$

for all $n \in N$. This shows that C_T is an isometry and hence C_T is quasinormal. Since T is one-to-one, it follows from the claim in Theorem 2 of Singh (1975b) that C_T has dense range. Hence C_T is unitary. This shows that C_T is normal.

REFERENCES

Halmos, P. R. (1951). Introduction to Hilbert Space and Theory of Spectral Multiplicity. Chelsea Publishing Company, New York.

Singh, R. K. (1972). Composition operators. Doctoral thesis, University of New Hampshire, U.S.A.

——— (1975a). Compact and quasinormal composition operators. *Proc. Am. math. Soc.*, **45**, 80–82. *MR*50 # 1043.

——— (1975b). Normal and Hermitian composition operators. *Proc. Am. math. Soc.*, **47**, 348–50. *MR*50 # 8153.

Singh, R. K., and Gupta, D. K. (1978). Normal and Invertible composition operators. *Bull. Austr. Math. Soc.*, **18**, 45–54.

Singh, R. K., and Komal, B. S. (1978). Composition operator on l^p and its adjoint. *Proc. Am. math. Soc.*, **70**, 21–26.

Singh, R. K., and Kumar, A. (1978). Characterization of invertible, unitary and normal composition operators. *Bull. Austr. Math. Soc.*, **19**, 81–95.