Indian J. pure appl. Math., 11(7) : 908-916, July 1980

NEWTONIAN ANALOGUE OF FORCE AND RELATIVISTIC DRAG ON A FREE PARTICLE IN THE GRAVITATIONAL FIELD OF A COMBINED KERR-NUT FIELD

T. SINGH AND R. B. S. YADAV

Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi 221005

(*Received* 23 *May* 1978; *after revision* 15 *June* 1979)

In the first part of the present paper the Newtonian analogue of force for the combined Kerr-NUT metric has been investigated. To the first order of approximation one component of the force vector corresponds to the Newtonian gravitational force. In the higher order of approximation the relativistic correction terms due to rotation and presence of gravitational analogue of a magnetic monopole are obtained. In the second part of the paper the motion of a freely falling body has been investigated. It is found that plane orbits are not possible. Also a radial fall is not possible and there is a rotational drag on the particle which has no Newtonian analogue.

1. INTRODUCTION

Newman, Tamburino and Unti (NUT) (1963) found a family of solutions of the Einstein's equations and Kerr obtained another solution both containing as a special case the Schwarzschild solution. Further, a generalization of the Kerr field, analogue to that proposed by Newman *et al.* was obtained by Demianski and Newman (1966) by means of a mathematical method involving a complex coordinate transformation. This combined Kerr-NUT solution has been further rederived and investigated by Reina and Treves (1975, 1976), Carmeli (1977) and Wei (1978). This metric (Reina and Treves 1976, Wei 1978) can be written in the Boyer-Lindquist coordinates in the form

$$\begin{aligned} ds^2 = & -\frac{(r^2 - 2mr - l^2 + a^2 \cos^2 \theta)}{\{r^2 + (l - a\cos\theta)^2\}} \{dt - [\{2a \sin^2\theta(l^2 + mr) \\ & - 2l\cos\theta(r^2 - 2mr - l^2 + a^2)\} \\ & \times (r^2 - 2mr - l^2 + a^2 \cos^2\theta)^{-1}]\, d\phi\}^2 \\ & + \frac{r^2 + (l - a\cos\theta)^2}{r^2 - 2mr - l^2 + a^2\cos^2\theta} \Big[(r^2 - 2mr - l^2 + a^2\cos^2\theta) \\ & \times \left(\frac{dr^2}{r^2 - 2mr - l^2 + a^2)} + d\theta^2\right) + (r^2 - 2mr - l^2 + a^2)\sin^2\theta\, d\phi^2\Big] \end{aligned} \quad ...(1.1)$$

where m, a and l are constants. The parameters m and a are the well known mass and angular velocity of the Kerr solution; the third parameter l is interpreted as the gravitational analogue of a magnetic monopole of mass l (Demianski and Newman 1966). The metric (1.1) reduces to the usual Boyer-Lindquist form (Boyer and Lindquist 1967) of the Kerr metric for $l = 0$ and to the NUT metric for $a = 0$. For $a = l = 0$, the metric (1.1) reduces to the well known Schwarzschild metric.

2. Newtonian Analogue of Force

In general relativity there is no invariant formulation of the concept of force as that in the Newtonian theory. Narlikar and Singh (1951) investigated the possibility of representing the analogue of Newtonian force in general relativity by means of certain invariants couched in terms of the field quantities against the background of an arbitrary flat substratum. The analogue of Newtonian force has been evaluated for Vaidya's metric by Mehra *et al.* (1969). Further Singh and Upadhyay (1971, 1972) have made similar investigations for Kerr and NUT metrics and they have found that, to the first order of approximation, the component of force vector is proportional to the Newtonian force.

The vectors R_i and S_i as defined by Narlikar and Singh (1951) and Singh and Upadhyay (1971) are

$$\text{(i)} \quad R_i = \Delta^{j}_{ji} = \frac{H_{,i}}{H}$$

$$\text{(ii)} \quad S_i = \Delta^{l}_{mn}\, g^{mn} g_{li}$$

$$= g^{mn} g_{mi,n} - \frac{H_{,i}}{H} \qquad \ldots(2.1)$$

where

$$H = \sqrt{g/\gamma}.$$

Here g is determinant of the metric tensor of general relativity g_{ij} and γ is the determinant of the flat substratum metric γ_{ij}. These vectors represent invariant features of relativistic gravitation and in the following we evaluate them for the line element (1.1).

For the line element (1.1) we have

$$g_{11} = \frac{\{r^2 + (l - a\cos\theta)^2\}}{(r^2 - 2mr - l^2 + a^2)},$$

$$g_{22} = \{r^2 + (l - a\cos\theta)^2\},$$

(*equation continued on p.* 910)

$$g_{33} = \frac{1}{(r^2 - 2mr - l^2 + a^2 \cos^2 \theta)}$$
$$\times \left[- \frac{\{2a \sin^2 \theta (l^2 + mr) - 2l \cos \theta (r^2 - 2mr - l^2 + a^2)\}^2}{\{r^2 + (l - a \cos \theta)^2\}} \right.$$
$$\left. + \{r^2 + (l - a \cos \theta)^2\} \times (r^2 - 2mr - l^2 + a^2) \sin^2 \theta \right],$$

$$g_{34} = g_{43} = \frac{\{2a \sin^2 \theta (l^2 + mr) - 2l \cos \theta (r^2 - 2mr - l^2 + a^2)\}}{\{r^2 + (l - a \cos \theta)^2\}},$$

$$g_{44} = - \frac{(r^2 - 2mr - l^2 + a^2 \cos^2 \theta)}{\{r^2 + (l - a \cos \theta)^2\}} \qquad \text{...(2.2)}$$

and

$$g^{11} = \frac{(r^2 - 2mr - l^2 + a^2)}{\{r^2 + (l - a \cos \theta)^2\}},$$

$$g^{22} = \frac{1}{\{r^2 + (l - a \cos \theta)^2\}},$$

$$g^{33} = - \frac{\{2mr - (r^2 - l^2 + a^2 \cos^2 \theta)\}}{\{r^2 + (l - a \cos \theta)^2\} (r^2 - 2mr - l^2 + a^2) \sin^2 \theta},$$

$$g^{34} = g^{43} = \frac{\{2a \sin^2 \theta (l^2 + mr) - 2l \cos \theta (r^2 - 2mr - l^2 + a^2)\}}{\{r^2 + (l - a \cos \theta)^2\} (r^2 - 2mr - l^2 + a^2) \sin^2 \theta},$$

$$g^{44} = \frac{\{2a \sin^2 \theta (l^2 + mr) - 2l \cos \theta (r^2 - 2mr - l^2 + a^2)\}^2}{(r^2 - 2mr - l^2 + a^2)(r^2 - 2mr - l^2 + a^2 \cos^2 \theta)\{r^2 + (l - a \cos \theta)^2\} \sin^2 \theta}$$
$$- \frac{\{r^2 + (l - a \cos \theta)^2\}}{(r^2 - 2mr - l^2 + a^2 \cos^2 \theta)},$$

$$g = - \{r^2 + (l - a \cos \theta)^2\}^2 \sin^2 \theta. \qquad \text{...(2.3)}$$

We take the corresponding flat metric as

$$ds^2 = (r^2 + a^2 \cos^2 \theta) \left[\frac{dr^2}{(r^2 + a^2)} + d\theta^2 \right]$$
$$+ (r^2 + a^2) \sin^2 \theta \, d\phi^2 - dt^2. \qquad \text{...(2.}$$

We then have

$$\gamma_{11} = \frac{(r^2 + a^2 \cos^2 \theta)}{r^2 + a^2},$$

$$\gamma_{22} = (r^2 + a^2 \cos^2 \theta),$$

$$\gamma_{33} = (r^2 + a^2) \sin^2 \theta,$$

(equation continued on p. 911)

$$\gamma_{44} = -1,$$

$$\gamma = -(r^2 + a^2 \cos^2 \theta)^2 \sin^2 \theta,$$

$$H = \frac{\{r^2 + (l - a \cos \theta)^2\}}{(r^2 + a^2 \cos^2 \theta)}. \qquad \text{...(2.5)}$$

We find that

$$R_1 = \frac{2rl(2a \cos \theta - l)}{(r^2 + a^2 \cos^2 \theta) \{r^2 + (l - a \cos \theta)^2\}},$$

$$R_2 = 2a \sin \theta \left\{ \frac{l - a \cos \theta}{r^2 + (l - a \cos \theta)^2} + \frac{a \cos \theta}{r^2 + a^2 \cos^2 \theta} \right\},$$

$$R_3 = R_4 = 0 \qquad \text{...(2.6)}$$

and

$$S_1 = \left[\frac{2r}{r^2 + (l - a \cos \theta)^2} - \frac{2(r - m)}{(r^2 - 2mr - l^2 + a^2)} - \frac{2rl(2a \cos \theta - l)}{(r^2 + a^2 \cos^2 \theta) \{r^2 + (l - a \cos \theta)^2\}} + \frac{r(2mr + l^2)}{(r^2 + a^2 \cos^2 \theta)(r^2 - 2mr - l^2 + a^2)} - \frac{r}{r^2 + a^2 \cos^2 \theta} - \frac{r \{2mr - (r^2 - l^2 + a^2 \cos^2 \theta)\} (r^2 + a^2)}{(r^2 - 2mr - l^2 + a^2)^2 (r^2 + a^2 \cos^2 \theta)} \right],$$

$$S_2 = -\tfrac{1}{2} \frac{a^2(2mr + l^2)^2 \sin 2\theta}{(r^2 + a^2)(r^2 + a^2 \cos^2 \theta)(r^2 - 2mr - l^2 + a^2)},$$

$$S_3 = S_4 = 0. \qquad \text{...(2.7)}$$

If we put $a = l = 0$ in (2.6) and (2.7), we get

$$R_i = [0, 0, 0, 0] \qquad \text{...(2.8)}$$

$$S_i = \left[\left\{ \frac{2m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1} \right\}, 0, 0, 0 \right] \qquad \text{...(2.9)}$$

which are the corresponding expressions of R_i and S_i for Schwarzschild external line element already evaluated by Narlikar and Singh (1951).

We now study the non-zero components of R_i and S_i in some detail.

The Component R_1

The component R_1 is given by

$$R_1 = \frac{4arl \cos \theta - 2rl^2}{(r^2 + a^2 \cos^2 \theta) \{r^2 + (l - a \cos \theta)^2\}}. \qquad \text{...(2.10)}$$

Equation (2.10) and the value of g^{11} give

$$R^1 = \frac{(r^2 - 2mr - l^2 + a^2)(4arl\cos\theta - 2rl^2)}{\{r^2 + (l - a\cos\theta)^2\}^2 (r^2 + a^2\cos^2\theta)}. \qquad ...(2.11)$$

To the first order of approximation $R_1 = R^1 = 0$, but to the second order of approximation eqns. (2.10) and (2.11) provide

$$R_1 = \frac{4al\cos\theta}{r^3} - \frac{2l^2}{r^3}, \qquad ...(2.12)$$

$$R^1 = \frac{4al\cos\theta}{r^3} - \frac{2l^2}{r^3}. \qquad ...(2.13)$$

When there is no rotation $a = 0$ and $R_1 = R^1 = -2l^2/r^3$. This term arises due to magnetic type monopole of mass l.

The Component R_2

The component R_2 is given by

$$R_2 = 2a\sin\theta \left\{\frac{l - a\cos\theta}{r^2 + (l - a\cos\theta)^2} + \frac{a\cos\theta}{r^2 + a^2\cos^2\theta}\right\}. \qquad ...(2.14)$$

Equation (2.14) and the value of g^{22} give

$$R^2 = \frac{2a\sin\theta}{r^2 + (l - a\cos\theta)^2}\left\{\frac{l - a\cos\theta}{r^2 + (l - a\cos\theta)^2} + \frac{a\cos\theta}{r^2 + a^2\cos^2\theta}\right\}. \qquad ...(2.15)$$

To the second order of approximation eqns. (2.14) and (2.15) give

$$R_2 = \frac{2al\sin\theta}{r^2} \qquad ...(2.16)$$

$$R^2 = \frac{2al\sin\theta}{r^4}. \qquad ...(2.17)$$

From eqns. (2.12), (2.13), (2.16) and (2.17) we arrive at the conclusion that the presence of such forces is only due to rotation and monopole mass. These forces have no counterpart in Newtonian theory and are of purely relativistic nature.

The Component S_1

To the second order of approximation, we have

$$S_1 = \frac{2m}{r^2} + \frac{4m^2}{r^3} + \frac{a^2\sin^2\theta}{r^3} - \frac{2l^2}{r^3} \qquad ...(2.18)$$

$$S^1 = \frac{2m}{r^2} + \frac{a^2\sin^2\theta}{r^3} - \frac{2l^2}{r^3}. \qquad ...(2.19)$$

In this case we find that upto the second order of approximation S_1 and S^1 both are composed of two type of forces. The first order term corresponds to the Newtonian force and the rest part gives the relativistic correction due to the rotation of the source and the presence of the monopole.

The Component S_2

The component S_2 is given by

$$S_2 = -\frac{1}{2}\frac{a^2 \sin 2\theta(2mr + l^2)^2}{(r^2 + a^2)(r^2 - 2mr - l^2 + a^2)(r^2 + a^2 \cos^2\theta)}. \quad ...(2.20)$$

Equation (2.19) and the value of g^{22} yield

$$S^2 = -\frac{1}{2}\frac{a^2 \sin 2\theta(2mr + l^2)^2}{(r^2 + a^2)(r^2 - 2mr - l^2 + a^2)(r^2 + a^2 \cos^2\theta)\{r^2 + (l - a\cos\theta)^2\}}. \quad ...(2.21)$$

To the second order of approximation of eqns. (2.20) and (2.21) give

$$S_2 = S^2 = 0.$$

When $a = 0$, we get the results due to Singh and Upadhyay (1971) for NUT metric [if we adopt the interpretation of NUT metric due to Bonnor (1969)]. For $l = 0$, we have the results of Singh and Upadhyay (1972) for Kerr metric. When both $a = 0$ and $l = 0$, we have the results of Narlikar and Singh (1951) for Schwarzschild exterior metric.

3. Motion of a Freely Falling Particle

In this section the motion of a free particle in the gravitational field of combined Kerr-NUT metric has been discussed. It has been found that the particle does not trace a plane orbit and instead the orbit is a torquous curve. Further a radial fall is also not possible and there is a rotational drag on the particle which has no Newtonian counterpart and is of relativistic nature. This effect in the case of slowly rotating bodies may not be much effective but in the case of fast rotating bodies it will produce an appreciable effect.

The usual equation of geodesic is

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0, \quad \alpha = 1, 2, 3, 4. \quad ...(3.1)$$

If the coordinate time $t = x^4$ is taken as the independent variable instead of the proper time s, the geodesic eqn. (3.1) can be expressed in the form

$$\frac{d^2x^\alpha}{dt^2} - \frac{dx^\alpha}{dt}\Gamma^4_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0. \quad ...(3.2)$$

For the metric (1.1) the geodesic equations obtained from (3.2) corresponding to $\alpha = 1, 2, 3$ lead to three differential equations. (We have omitted these equations as they are quite lengthy).

If we choose the coordinates so that the particle moves initially in the plane $\theta = \pi/2$, then $(d^2\theta/dt^2) \neq 0$ which implies that the motion is not confined to this plane and the path will be a torquous curve.

Now if we consider a test particle which satisfies the conditions $\theta = \pi/2$, $(d\theta/dt) = 0$ and $\phi = 0$, $(d\phi/dt) = 0$ when $t = 0$, then we find $(d^2\phi/dt^2) \neq 0$. Hence the particle would not continue to move in radial direction and the geodesic eqns. (3.2) corresponding $\alpha = 1$ and $\alpha = 3$ reduce to

$$\begin{aligned} \ddot{r} &- [2\{mr^2(r^2 - 2mr)(r^2 + a^2) + l^2(-5r^2l^2m \\ &- 5mr^4 + 2m^2l^2r + ml^4 + a^2l^2m + 2r^5 + 2a^2r^3 \\ &- 2rl^4 - 2a^2l^2r + 2m^2a^2r)\}/\{(r^2 + l^2)^2 (r^2 \\ &- 2mr - l^2 + a^2)(r^2 - 2mr - l^2)\} - \{m(l^2 - r^2) \\ &+ r(a^2 - 2l^2)\}/\{(r^2 + l^2)(r^2 - 2mr - l^2 + a^2)\}]\,\dot{r}^2 \\ &- [(r^2 - 2mr - l^2 + a^2)(ml^2 - mr^2 - 2rl^2)]/(r^2 + l^2)^3 = 0 \qquad \text{...(3.3)} \end{aligned}$$

$$\ddot{\phi} - 2a\,(mr^2 - ml^2 + 2rl^2)/\{(r^2 + l^2)^2 (r^2 - 2mr - l^2 + a^2)\}\,\dot{r} = 0 \qquad \text{...(3.4)}$$

If we consider the second order of approximation of eqn. (3.3), we get

$$\ddot{r} = -\frac{m}{r^2}\left(1 - \frac{2m}{r}\right) + \frac{3m}{r^2}\left(1 + \frac{2m}{r}\right)\dot{r}^2 - \frac{a^2}{r^3}\dot{r}^2 + \frac{6l^2}{r^3}\dot{r}^2 - \frac{2l^2}{r^3}. \qquad \text{...(3.5)}$$

In (3.5) the first part gives the Newtonian force, second part gives the repulsive force depending upon the square of the radial velocity as pointed out by Singh and Pandey (1960) and the remaining part gives the relativistic correction due to the rotation of the source and the presence of the monopole.

To the second order of approximation, the geodesic equations, obtained from (3.2) under the conditions $\theta = (\pi/2)$, $\dot{\theta} = 0$ when $t = 0$, may be written as

$$\begin{aligned} \ddot{r} - r\dot{\phi}^2 = &-\frac{m}{r^2}\left(1 - \frac{2m}{r}\right) + \frac{3m}{r^2}\left(1 + \frac{2m}{r}\right)\dot{r}^2 \\ &- 2m\dot{\phi}^2 + \frac{6am}{r^2}\dot{r}^2\dot{\phi} - \frac{a^2}{r^3}\dot{r}^2 + \frac{a^2}{r}\dot{\phi}^2 \\ &- \frac{2am}{r^2}\dot{\phi} - \frac{2l^2}{r}\dot{\phi}^2 + \frac{6l^2}{r^3}\dot{r}^2 - \frac{2l^2}{r^3} \qquad \text{...(3.6)} \end{aligned}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = \frac{6am}{r}\,\dot{r}\dot{\phi}^2 + \frac{2am}{r^3}\,\dot{r} + \frac{2m}{r}\left(1 + \frac{2m}{r}\right)\dot{r}\dot{\phi} + \frac{2a^2}{r^2}\,\dot{r}\dot{\phi} + \frac{6l^2}{r^2}\,\dot{r}\dot{\phi}. \qquad ...(3.7)$$

From (3.6) it is clear that in addition to Newtonian force there is a repulsive force depending upon the square of the radial velocity and an attractive force proportional to the square of the angular velocity together with some relativistic correction terms. The appearance of these correction terms is due to the rotation of the source and the monopole of mass l and they have no counterpart in the Newtonian theory.

The results of this section will specialize to those of Singh and Ram (1972) when $a = 0$, $l \neq 0$ (i.e. for NUT metric) whereas for $a \neq 0$, $l = 0$ (i.e. Kerr metric) we get the results of Singh and Upadhyay (1973). When $a = l = 0$ we have the results of Singh and Pandey (1960) for Schwarzschild metric.

Acknowledgement

The authors are thankful to the State Council of Science and Technology, U.P., for financial support. Thanks are also due to the referee for some useful suggestions.

References

Bonnor, W. B. (1969). A new interpretation of the NUT metric in general relativity. *Proc. Camb. phil. Soc.*, **66**, 145–51.

Boyer, R. H., and Lindquist, R. W. (1967). Maximal analytic extension of the Kerr metric. *J. Math. Phys.*, **8**, 265–71.

Carmeli, M. (1977). Group Theory and General Relativity. McGraw-Hill Book Co., Inc., New York, p. 275.

Demianski M., and Newman, E. T. (1966). A combined Kerr-NUT solution of the Einstein field equations. *Bull. Acad. Pol. Sci. Math. Astron. Phys.*, **14**, 653–57.

Kerr, R. P. (1963). Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, **11**, 237–38.

Mehra, A. L., Vaidya, P. C., and Kushwaha, R. S. (1969). Expansion of a radiating metric. *Acta Phys. Acad. Scient. Hungarica*, **26**, 339.

Narlikar, V. V., and Singh, K. P. (1951). The role of the three-index symbols in relativity. *Proc. natn. Inst. Sci.*, India, **17**, 311.

Newman, E. T., Tamburino, L., and Unti, T. (1963). Empty space generalization of the Schwarzschild metric. *J. Math. Phys.*, **4**, 915–23.

Reina, C., and Treves, A. (1975). NUT-like generalization of the axisymmetric gravitational fields. *J. Math. Phys.*, **16**, 934–35.

——— (1976). Axisymmetric gravitational fields. *Gen. Rel. Grav.*, **7**, 817–38.

Rosen, N. (1940). General relativity and flat space I. *Phys. Rev.*, **57**, 147–50.

Singh, K. P., and Pandey, S. N. (1960). A freely falling observer in a relativistic gravitational field. *Proc. natn. Inst. Sci.*, India, **26**, 694–98.

Singh, K. P., and Sri Ram (1972). Rotational effect on a test particle in the gravitational field of NUT metric. *Acad. Prog. Math. (India)*, **6**, 1–3.

Singh, K. P., and Upadhyay, R. C. (1971). Newtonian analogue of force in some gravitational fields. *Acad. Prog. Math. (India)*, **5**, 76–81.

——————— (1972). Newtonian analogue of force in Kerr metric. *Indian J. Phys.*, **46**, 172–76.

——————— (1973). Relativistic drag on a free particle in the gravitational field of Kerr metric. *Acad. Prog. Math. (India)*, **7**, 1–4.

Wei, M. S. (1978). A generalization of the Kerr-Taub-NUT solution. *Il. Nuovo Cimento*, **44B**, 405–408.