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A PLANE SYMMETRIC UNIVERSE

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A plane symmetric perfect fluid cosmological model has been derived. Various physical and geometrical properties have been discussed.

1. INTRODUCTION

In recent years astronomical observations have tended to confirm a degree of spatial anisotropy in the large scale behaviour of the universe. Homogeneous models with spatial anisotropy have been widely studied. Particular examples of such models are the plane symmetric cosmological models in which the matter distribution is that of a perfect fluid. A plane symmetric cosmological model has been derived by Singh and Singh (1968). Singh and Abdussattar (1973) also constructed a plane symmetric cosmological model describing a perfect fluid distribution. In this paper a plane symmetric cosmological model for perfect fluid has been derived. The model represents an expanding, shearing but non-rotating fluid flow which is also geodesic. The expression for the generalized Doppler effect in the model has been given.

2. DERIVATION OF THE LINE-ELEMENT

We consider the plane symmetric metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2 \qquad ...(2.1)$$

where A, B, C are functions of t alone. This ensures that the model is spatially homogeneous. The energy-momentum tensor for perfect fluid distribution is given by

$$T_i^k = (\epsilon + p)\, V_i V^k + p\, g_i^k \qquad ...(2.2)$$

together with

$$V_i V^i = -1 \qquad ...(2.3)$$

p being the isotropic pressure, ϵ the density and V^i the flow vector satisfying (2.3). We assume the coordinates to be comoving so that

$$V^1 = V^2 = V^3 = 0 \text{ and } V^4 = 1/A.$$

The field equations

$$R_i^k - \tfrac{1}{2} R\, \delta_i^k + \Lambda \delta_i^k = -\, 8\pi\, T_i^k \qquad ...(2.4)$$

for the line element (2.1) are as follows:

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4 C_4}{BC} + \frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC}\right] - \Lambda = 8\pi p \qquad ...(2.5)$$

$$\frac{1}{A^2}\left[-\frac{C_{44}}{C} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda = 8\pi p \qquad ...(2.6)$$

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda = 8\pi p \qquad ...(2.7)$$

$$\frac{1}{A^2}\left[\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC}\right] + \Lambda = 8\pi\epsilon. \qquad ...(2.8)$$

The suffix 4 after the symbols A, B, C denote ordinary differentiation with respect to t. These are four equation in five unknowns A, B, C, ϵ and p. For complete solution of eqns. (2.5) — (2.8) we need an extra condition. However, the set of equations is so simple that without using any assumption it is possible by performing a couple of integration to express A, B and C in terms of one unknown function. We shall therefore, not take any extra condition. From (2.6) and (2.7) we get

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = 0 \qquad ...(2.9)$$

which on integration gives

$$B = C\,[\,\textstyle\int (K/C^2)\, dt + N] \qquad ...(2.10)$$

K and N being constants of integration. From eqns. (2.5) and (2.6) we get

$$\left[\frac{A_4}{A}\right]_4 + \frac{A_4}{A}\left[\frac{B_4}{B} + \frac{C_4}{C}\right] = \frac{B_{44}}{B} + \frac{B_4 C_4}{BC}. \qquad ...(2.11)$$

Equation (2.11) on integration gives

$$A = MB \exp\,[\,\textstyle\int (L/BC)\, dt] \qquad ...(2.12)$$

where L and M are constants of integration. Consequently, the line-element (2.3) after a suitable transformation of coordinates takes the form

$$ds^2 = C^2\alpha^2 \exp\,[\,\textstyle\int (2L/MC^2\alpha)\, dT]\,(dX^2 - dT^2) + C^2(\alpha^2 dY^2 + dZ^2) \qquad ...(2.13)$$

where

$$\alpha = [\,\textstyle\int (K/MC^2)\, dT + N].$$

3. Some Physical and Geometrical Features

The distribution in the model (2.13) is given by

$$8\pi p = \frac{1}{C^2\alpha^2 \exp\left[\int (2L/\alpha MC^2)\, dT\right]} \left[\frac{\alpha_T}{\alpha^2 C^2} + \frac{2C_T}{\alpha C^3} - \left(\frac{\alpha_T}{\alpha}\right)_T - \left(\frac{C_T}{C}\right)_T - \frac{C_{TT}}{C}\right] - \Lambda \qquad \ldots(3.1)$$

and

$$8\pi\epsilon = \frac{1}{C^2\alpha^2 \exp\left[\int (2L/\alpha MC^2)\, dT\right]} \left[3\left(\frac{C_T}{C}\right)^2 + \frac{4\alpha_T C_T}{\alpha C} + \left(\frac{\alpha_T}{\alpha}\right)^2 + \frac{L}{M\alpha C^2}\left(2\frac{C_T}{C} + \frac{\alpha_T}{\alpha}\right)\right] + \Lambda \qquad \ldots(3.2)$$

where $C_T = \frac{dC}{dT}$ and $\alpha_T = \frac{d\alpha}{dT} = \frac{K}{MC^2}$.

The model has to satisfy the reality conditions (Ellis 1971)

(i) $(\epsilon + p) > 0$

and

(ii) $(\epsilon + 3p) > 0$

which impose restrictions on the time during which the model exists. The flow vector V^i of the distribution is given by

$$V^4 = \frac{1}{C\alpha} \exp\left[\int (-L/M\alpha C^2)\, dT\right],\quad V_4 = -C\alpha \exp\left[\int (L/M\alpha C^2)\, dT\right].$$

Clearly $V^i_{;j} V^j = 0$ so that the flow is geodesic.

The motion of a test particle in the model is governed by the geodesics given by

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \qquad \ldots(3.3)$$

From the equations of the geodesic we conclude that if a particle is initially at rest in the model it would remain permanently at rest. Following the method outlined by Tolman (1962) the red shift in the model is given by

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{C\alpha \exp\left[\int (L/M\alpha C^2)\, dT\right]}{\left[C^2\alpha^2 \exp\left\{\int (2L/M\alpha C^2)\, dT\right\} - U^2\right]^{1/2}} \frac{\phi(T_1) + U_Z}{\phi(T_2)} \qquad \ldots(3.4)$$

where $\phi(T) = \alpha \exp\left[\int (L/M\alpha C^2)\, dT\right]$, U is the velocity of the source at the time of emission and U_Z the Z-component of the velocity. The expressions for expansion Θ, rotation ω and shear σ_{ij} calculated for the flow vector V^i are given by

$$\Theta = \frac{1}{C\alpha \exp\left[\int (L/M\alpha C^2)\, dT\right]} \left[\frac{3C_T}{C} + \frac{2\alpha_T}{\alpha} + \frac{L}{M\alpha C^2}\right] \quad \ldots(3.5)$$

$$\omega = 0$$

and

$$\left.\begin{aligned} \sigma_{11} &= \tfrac{1}{3} C \exp\left[\int (L/M\alpha C^2)\, dT\right] \left[\alpha_T + \frac{2L}{MC^2}\right] \\ \sigma_{22} &= \tfrac{1}{3} C \exp\left[\int (-L/M\alpha C^2)\, dT\right] \left[\alpha_T - \frac{L}{MC^2}\right] \\ \sigma_{33} &= -\tfrac{1}{3} \frac{C}{\alpha^2} \exp\left[\int (-L/M\alpha C^2)\, dT\right] \left[2\alpha_T + \frac{L}{MC^2}\right] \end{aligned}\right\} \quad \ldots(3.6)$$

the other components of the shear tensor σ_{ij} being zero. The surviving components of the conformal curvature tensor for the line-element (2.13) are given by

$$C^{14}_{14} = C^{23}_{23} = \frac{-1}{3C^2\alpha^2 \exp\left[\int (2L/\alpha MC^2)\, dT\right]} \left[\left(\frac{\alpha_T}{\alpha}\right)_T - \left(\frac{L}{M\alpha C^2}\right)_T - \frac{\alpha_T C_T}{\alpha C}\right] \quad \ldots(3.7)$$

$$C^{12}_{12} = C^{34}_{34} = \frac{1}{6C^2\alpha^2 \exp\left[\int (2L/\alpha MC^2)\, dT\right]} \times \left[\frac{\alpha_{TT}}{\alpha} - \frac{4\alpha_T^2}{\alpha^2} - \frac{2\alpha_T C_T}{\alpha C} + \left(\frac{L}{M\alpha C^2}\right)_T - \frac{3L\alpha_T}{M\alpha^2 C^2}\right]$$

$$C^{13}_{13} = C^{24}_{24} = \frac{1}{6C^2\alpha^2 \exp\left[\int (2L/\alpha MC^2)\, dT\right]} \times \left[\frac{\alpha_{TT}}{\alpha} + \frac{2\alpha_T^2}{\alpha^2} + \frac{4\alpha_T C_T}{\alpha C} + \left(\frac{L}{M\alpha C^2}\right)_T + \frac{3L\alpha_T}{M\alpha^2 C^2}\right].$$

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