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ON SINGULAR SURFACES OF ORDER ONE IN DISSOCIATING GASES

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In this paper the authors study the growth and decay of sonic discontinuities in an ideal dissociating gas by taking the entropy and frozen velocity of sound as thermodynamical variables. The speed of the sonic wave is shown to be equal to the frozen velocity of sound and a scalar defined on the wave surface is shown to decrease as the degree of dissociation increases. Some results pertaining to vorticity, entropy and curvature of streamlines are also obtained.

INTRODUCTION

By taking the pressure p and density ρ of an ideal gas as thermodynamical variables, Thomas (1957) discussed the growth and decay of sonic discontinuities. Kaul (1961) discussed the same problem but took the entropy S and local velocity of sound C as the thermodynamical variables in place of p and ρ. Verma and Srivastava (1977) obtained the effects of magnetic field on the flow and discussed singular surfaces of order one in ideal gases.

Here, we have considered an ideal dissociating gas and taken entropy S and 'frozen velocity' of sound C as thermodynamical variables. It has been obtained that the speed of the sonic wave equals the frozen velocity of sound and the magnitude of a scalar λ (defined on the surface in the following section) decreases as the degree of dissociation increases. Some results pertaining to vorticity, derivatives of entropy and curvature of streamlines have also been obtained.

BASIC EQUATIONS

The equation of continuity for the atom mass species of Lighthill's model of an ideal dissociating gas can be written as (Lighthill 1957)

$$\frac{\partial \alpha}{\partial t} + u_i \alpha_{,i} = w \qquad ...(1)$$

where α is the degree of dissociation.

The hydrodynamical equations referred to a system of rectangular coordinates x_i (with comma denoting parial differentiation with respect to these coordinates) are

$$\frac{\partial \rho}{\partial t} + u_i \rho_{,i} + \rho u_{i,i} = 0 \qquad ...(2)$$

$$\rho \frac{\partial u_i}{\partial t} + \rho n_j u_{i,j} + p_{,i} = 0 \qquad \text{...(3)}$$

$$\frac{\partial S}{\partial t} + u_i S_{,i} = 0 \qquad \text{...(4)}$$

where

$$S = 3 \log T + \alpha(1 - 2 \log \alpha) - (1 - \alpha) \log (1 - \alpha) - (1 + \alpha) \log \rho + \text{const.} \qquad \text{...(5)}$$

and u_i, ρ, p and S are components of velocity, denisty, pressure and entropy, and

$$w = \frac{k_r(1 + \alpha)\rho^2}{m_a^2} \left[\frac{\alpha_e^2}{1 - \alpha_e^2} (1 - \alpha^2) - \alpha^2 \right]$$

k_r being the recombination rate, m_a the mass of atom and α_e, the equilibrium degree of dissociation. For a dissociating gas the 'frozen' velocity of sound C as given by Bhutani and Rama Shanker (1970) is

$$C^2 = \frac{4 + \alpha}{3} \frac{p}{\rho}. \qquad \text{...(6)}$$

Taking the entropy S and 'frozen velocity' of sound C as thermodynamical variables in place of pressure p and density ρ, eqns. (2) and (3) assume the following alternative forms:

$$\frac{6}{(1 + \alpha) C} \left(\frac{\partial C}{\partial t} + u_i C_{,i} \right) + u_{i,i} + \left[-3 \frac{(5 + 2\alpha)}{(4 + \alpha)(1 + \alpha)} + \log \frac{(1 - \alpha)}{\alpha^2} + \frac{1}{(1 + \alpha)} \left\{ S - 3 \log \frac{3C^2}{(4 + \alpha)(1 + \alpha)} - \alpha(1 - 2 \log \alpha) + (1 - \alpha) \log (1 - \alpha) - \text{const.} \right\} \times \left(\frac{\partial \alpha}{\partial t} + u_i \alpha_{,i} \right) \right] = 0 \qquad \text{...(7)}$$

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} + \frac{3C^2}{(4 + \alpha)(1 + \alpha)} \left[- S_{,i} + 2(4 + \alpha) \frac{C_{,i}}{C} + \left(- \frac{4 + \alpha}{1 + \alpha} + \log \frac{1 + \alpha}{\alpha^2} + \frac{1}{(1 + \alpha)} \left\{ S - 3 \log \frac{3C^2}{(4 + \alpha)(1 + \alpha)} - \alpha(1 - 2 \log \alpha) + (1 - \alpha) \log (1 - \alpha) + \text{const.} \right\} \right) \alpha_{,i} \right] = 0. \qquad \text{...(8)}$$

If $\Sigma(t)$ denotes a moving singular surface of order one, relative to at least one of the first derivatives of S, C or u_i and a square bracket [ ] denotes the discontinuity or jump in a quantity across $\Sigma(t)$, the compatibility conditions of the first order are (Thomas 1957)

$$\left.\begin{aligned} [u_{i,k}] &= \lambda_i \nu_k, \left[\frac{\partial u_i}{\partial t}\right] = -\lambda_i G \\ [C_{,i}] &= \xi \nu_i, \left[\frac{\partial C}{\partial t}\right] = -\xi G \\ [S_{,i}] &= \zeta \nu_i, \left[\frac{\partial S}{\partial t}\right] = -\zeta G \end{aligned}\right\} \quad \ldots(9)$$

where ν_i are the components of the unit normal vector ν to $\Sigma(t)$, G denotes the normal velocity of the discontinuity surface and λ_i, ξ and ζ are suitable functions defined over the surface $\Sigma(t)$.

Also, it is supposed that $[\alpha] = 0$. Taking jumps of eqns. (1), (7), (8) and (4) and using compatibility conditions we obtain

$$(G - U_n)[\alpha_{,i}]\nu_i = 0 \quad \ldots(10)$$

$$\frac{6}{(1+\alpha)}(G - U_n)\xi = C\lambda_i \nu_i \quad \ldots(11)$$

$$(U_n - G)\lambda_i + \frac{3C^2}{(4+\alpha)(1+\alpha)}\Big[-\zeta\nu_i + \frac{2(4+\alpha)}{C}\xi\nu_i + \Big(-\frac{4+\alpha}{1+\alpha} + \log\frac{1+\alpha}{\alpha^2} + \frac{1}{(1+\alpha)}\Big\{S - 3\log\frac{3C^2}{(4+\alpha)(1+\alpha)} - \alpha(1 - 2\log\alpha) + (1-\alpha)\log(1-\alpha)\Big\}\Big)[\alpha_{,i}]\Big] = 0 \quad \ldots(12)$$

$$\zeta(U_n - G) = 0 \quad \ldots(13)$$

Equation (10) implies either,

(i) $G - U_n = 0$ and $[\alpha_{,i}] = 0$

or

(ii) $G - U_n = 0$ and $[\alpha_i] \neq 0$

or

(iii) $G - U_n \neq 0$ and $[\alpha_{,i}] = 0$

and eqn. (13) implies either,

(i) $U_n - G = 0$ and $\zeta = 0$

or

(ii) $U_n - G = 0$ and $\zeta \neq 0$

or

(iii) $U_n - G \neq 0$ and $\zeta = 0$.

Combining the above two conditions we get either,

(i) $U_n - G = 0, \zeta = 0$ and $[\alpha_{,i}] = 0$

or

(ii) $U_n - G = 0, \zeta \neq 0$ and $[\alpha_{,i}] \neq 0$

or

(iii) $U_n - G \neq 0, \zeta = 0$ and $[\alpha_{,i}] = 0$.

Case (i)

If $U_n - G = 0$, $\zeta = 0$ and $[\alpha_{,i}] = 0$ eqn. (11) yields,

$$\lambda_i \nu_i = 0 \quad \text{...(14)}$$

it being, assumed that the 'frozen velocity' of sound C is not zero on $\Sigma(t)$. Equation (14) shows that the divergence of the velocity is continuous. Also from eqn. (12) we get

$$\xi \nu_i = 0.$$

Thus the surface $\Sigma(t)$ is such that

$$U_n - G = 0, \ \zeta = 0 \ [\alpha_{,i}] = 0, \ \lambda_i \nu_i = 0 \ \text{and} \ \xi = 0.$$

It is obvious that all λ_i cannot vanish over the surface $\Sigma(t)$ since otherwise all the first order derivatives of u_i, S, C and α would become continuous across the surface and then $\Sigma(t)$ would not be a singular surface of order one. Thus the divergence of the velocity and the first order derivatives of S, C and α are continuous across $\Sigma(t)$. The jump in the vorticity components w^k is given by

$$[w^k] = \epsilon^{ijk} [u_{j,i}] = \epsilon^{ijk} \lambda_j \nu_i \quad \text{...(15)}$$

where ϵ^{ijk} is the usual skew-symmetric permutation tensor showing thereby that w^k must be discontinuous over $\Sigma(t)$. Differentiating eqn. (4) with respect to x_j and remembering that the first order derivatives of entropy are continuous across $\Sigma(t)$ for the case under consideration, we get

$$\left[\frac{\partial^2 S}{\partial x_j \partial t}\right] + u_i [S_{,ij}] + S_{,i} [u_{i,j}] = 0. \quad \text{...(16)}$$

Further the second order compatibility conditions (Thomas 1957) for S become

$$[S_{,ij}] = \bar{\zeta} \nu_i \nu_j, \left[\frac{\partial^2 S}{\partial x_j \partial t}\right] = - G \bar{\zeta} \nu_j, \left[\frac{\partial^2 S}{\partial t^2}\right] = G^2 \bar{\zeta} \quad \text{...(17)}$$

where $\bar{\zeta}$ is some suitable scalar defined on $\Sigma(t)$. Equations (16) and (17) give

$$(G - U_n) \bar{\zeta} \nu_j = \lambda_i S_{,i} \nu_j. \quad \text{...(18)}$$

Since in the case considered above $G - U_n = 0$, eqn. (18) gives

$$\lambda_i S_{,i} \nu_j = 0.$$

That is,

$$\lambda_i S_{,i} = 0 \quad (\text{since all } \nu_j \neq 0). \qquad \text{...(19)}$$

Equation (19) therefore imposes a further restriction, being

$$\lambda_i \nu_i = 0$$

as given by (14).

From this it follows that the type of singular surface given by the case (i) is possible in one dimensional unsteady flow of an ideal dissociating gas.

Case (ii)

If $U_n - G = 0$, $\zeta \neq 0$ and $[\alpha_{,i}] \neq 0$, from eqn. (11) we get

$$\lambda_i \nu_i = 0,$$

it being assumed that C is not zero anywhere on $\Sigma(t)$. Also from eqn. (12) we get

$$\frac{2(4+\alpha)}{C}\xi\nu_i + \Big[-\frac{4+\alpha}{1+\alpha} + \log\frac{1+\alpha}{\alpha^2} + \frac{1}{1+\alpha}$$
$$\times\Big\{S - 3\log\frac{3C^2}{(4+\alpha)(1+\alpha)} - \alpha(1-2\log\alpha) + (1-\alpha)\log(1-\alpha)\Big\}\Big]$$
$$\times [\alpha_{,i}] = \zeta\nu_i.$$

Thus, across such a singular surface of order one, the first order partial derivatives of entropy, 'frozen velocity' of sound and degree of dissociation are discontinuous and the jumps in these are connected by the relation

$$\frac{[S_{,i}]}{[\alpha_{,i}]} - \frac{2(4+\alpha)}{C}\frac{[C_{,i}]}{[C_{,i}]} = -\frac{4+\alpha}{1+\alpha} + \log\frac{1+\alpha}{\alpha^2}$$
$$+\frac{1}{1+\alpha}\Big\{S - 3\log\frac{3C^2}{(4+\alpha)(1+\alpha)} - \alpha(1-2\log\alpha)$$
$$+(1-\alpha)\log(1-\alpha)\Big\}. \qquad \text{...(20)}$$

The divergence of the velocity is continuous across $\Sigma(t)$, the first order partial derivative of velocity need not be continuous. However, if all λ_i are zero then the vorticity is also continuous otherwise the vorticity will be discontinuous.

In both the cases discussed above the divergence of velocity is continuous across $\Sigma(t)$ and vorticity vector is discontinuous across $\Sigma(t)$. They differ only in the sense that the first order derivatives of entropy and degree of dissociation are continuous in the first case whereas they are discontinuous in the second case. Also, $U_n - G = 0$ in both the cases. This shows that the surface velocity G is equal to the normal component of the velocity of the dissociating gas.

Case (iii)

If $U_n - G \neq 0$, $\zeta = 0$ and $[\alpha,_i] = 0$, then eqn. (12) becomes

$$(U_n - G)\lambda_i + \frac{6C}{1+\alpha}\xi\nu_i = 0. \qquad \ldots(21)$$

Multiplying it by ν_i and using summation convention we get

$$(G - U_n)\lambda = \frac{6C}{1+\alpha}\xi \qquad \ldots(22)$$

where $\lambda_i\nu_i = \lambda$ and $\nu_i\nu_i = 1$.

Also eqn. (11) can be written as

$$\frac{6}{1+\alpha}(G - U_n)\xi = C\lambda. \qquad \ldots(23)$$

From eqns. (22) and (23) we get,

$$G - U_n = C \qquad \ldots(24)$$

and

$$\lambda = \frac{6}{1+\alpha}\xi. \qquad \ldots(25)$$

Equation (24) shows that the velocity of the sonic wave is equal to the 'frozen velocity' of sound and eqn. (25) shows that the value of λ decreases with the increasing degree of dissociation. Also from (21), (24) and (25) we get

$$\lambda_i = \lambda\nu_i \qquad \ldots(26)$$

which shows that the quantities λ_i can be obtained from a single scalar λ defined over $\Sigma(t)$. Thus the surface $\Sigma(t)$ is such that

$$G - U_n = C,\ \zeta = 0,\ \lambda_i = \lambda\nu_i,\ \lambda = \frac{6}{1+\alpha}\xi\ [\alpha,_i] = 0.$$

This singular surface is called sonic wave moving with a speed C (frozen velocity of sound). The first order derivative of entropy is continuous, although the ratio of the two scalars λ and ξ which gives jumps in the first derivative of velocity of the gas and 'frozen velocity' of sound is a function of the degree of dissociation α only and hence is a constant for a particular gas at a particular temperature.

From eqn. (25) it follows that

$$\frac{\delta\lambda}{\delta t} = \frac{6}{1+\alpha}\frac{\delta\xi}{\delta t} \qquad \text{...(27)}$$

where δ is time derivative defined by Thomas (1957). This direct proportionality between λ and ξ and between $\delta\lambda/\delta t$ and $\delta\xi/\delta t$ which has been obtained by taking the thermodynamical variables for the gas as S and C in place of p and ρ would yield considerable simplifications in discussing the growth and decay of a sonic wave propagating in an ideal dissociating gas where the thermodynamical variables are not costant.

The Vorticity Components

Using eqn. (26) the vorticity components w^k can be written as

$$[w^k] = \epsilon^{ijk}[u_{j,i}] = \epsilon^{ijk}\lambda_j\nu_i = \epsilon^{ijk}\nu_j\nu_i = 0.$$

This shows that the vorticity vector is continuous across the sonic wave whereas the first derivative of velocity is discontinuous across it.

The behaviour of second derivatives of entropy and curvature of the stream lines in an ideal dissociating gas is the same as in the case of ordinary gases and hence it is not discussed here.

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