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WEAK DISCONTINUITIES IN MAGNETOHYDRODYNAMICS

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Transport equations governing the propagation of weak discontinuities in an unsteady motion involving plane and cylindrically symmetric flow of a plasma have been derived by using the singular surface theory. The plasma is assumed to be an ideal gas with infinite electrical conductivity, and to be permeated by a magnetic field which is orthogonal to the trajectories of the gas particles. The exact predictions of the true non-linear progress of the flow and field gradients at the wave front are made and an expression for the time t_c taken to form a shock wave is obtained. It is concluded that the magnetic field effects cause the compression wave to steepen more swiftly than it does in an ordinary atmosphere. In the limit of zero megnetic field, the previously known results of ordinary gasdynamics are recovered.

INTRODUCTION

It is well understood that in the simple wave resulting from the positive acceleration of a piston into a (one-dimensional) gas at rest a shock wave must occur (Courant and Friedricks 1948). It is not, however, immediately obvious that such a breakdown of continuity in a compressive wave must necessarily occur when the motion has a spherical or cylindrical symmetry. Kuo (1946), Thomas (1957b) and Pack (1960) examined this problem in ordinary gasdynamics and showed that the cylindrically or spherically symmetrical motions retain the tendency to the shock formation and the spreading out of the wave shall not offset the tendency of the shock formation.

The purpose of this paper is to examine what may occur to the tendency of shock formation in an unsteady motion involving plane or cylindrically symmetrical flow of a plasma. The plasma is assumed to be an ideal gas with infinite electrical conductivity, and to be permeated by a uniform magnetic field which is orthogonal to the trajectories of gas particles (in the cylindrical case along the axis of symmetry). Using the compatibility conditions derived by Thomas (1957a), the speed of propagation of weak discontinuities and the transport equations governing their propagation have been obtained. Assuming the state ahead of the wave to be uniform and known, the exact predictions of the true non-linear progress of the flow and field gradients at the wave front are made and an expression for the time t_c taken to form a shock wave is obtained. Emphasis, however, is on a description of the gasdynamic phenomenon involved rather than on numerical results. It is shown that a compression wave with a plane or cylindrical wave front steepens up into a shock wave at a finite time whereas an expansion wave decays monotonically in time. It is concluded that the magnetic field effects cause the compression wave to steepen more swiftly than it does in an ordinary atmosphere. In the limit of zero magnetic field, the results obtained by Thomas (1957b) and Pack (1960) are recovered.

In this context, the works of Rarity (1967), Coleman and Gurtin (1967) and Becker (1970) on the breakdown of continuous motions in one-dimensional fluid flow exhibiting mechanical dissipation through the relaxation of internal state variables are worth mentioning.

Basic Equations and the Compatibility Conditions

Under the assumptions made, all the quantities are functions only of the radius r and time t, and the motion of the gas is governed by the following equations

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial r} + u \frac{\partial \rho}{\partial r} + \frac{\alpha \rho u}{r} = 0 \text{ (continuity)} \qquad \ldots(1)$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial r} + \frac{\partial p}{\partial r} + \mu H \frac{\partial H}{\partial r} = 0 \text{ (momentum)} \qquad \ldots(2)$$

$$\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial r} + H \frac{\partial u}{\partial r} + \frac{\alpha u H}{r} = 0 \text{ (induction)} \qquad \ldots(3)$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + \gamma p \left(\frac{\partial u}{\partial r} + \frac{\alpha u}{r} \right) = 0 \text{ (entropy)} \qquad \ldots(4)$$

p is the gas pressure, ρ the density, u the radial velocity and H the magnetic field orthogonal to the trajectories of the gas particles (in the cylindrical case along the axis of symmetry). The specific heat γ is assumed to be constant and the magnetic permeability μ is taken to be that of free space. The coefficient $\alpha = 0, 1$ refers to the case of plane and cylindrical symmetry respectively. Let $r = R(t)$ or for brevity $S(t)$ denote the weak discontinuity surface across which the flow and field parameters p, ρ, u and H are essentially continuous but discontinuities in their derivatives are permitted. The flow and field parameters ahead of the discontinuity surface are assumed to be uniform, as a consequence of which the successive position of $S(t)$ at different instants form a family of parallel surfaces with straight lines as their orthogonal trajectories (Thomas 1961). Thus given the surface at $t = t_0$ [$S(t_0)$ say], the position of the surface at any time $t > t_0$ can be determined by measuring the distance traversed by the wave along the normals to $S(t_0)$. If $\sigma (= R - R_0)$ denotes the distance measured from $S(t_0)$ along the normal trajectories then we have $\sigma = G(t - t_0)$, where G is the speed of propagation of the wave; R_0 being the value of R at $t = t_0$. In this case, the first and second order compatibility conditions for a singular surface derived by Thomas (1957a) reduce to

$$\left[\frac{\partial z}{\partial t}\right] = -AG \qquad ...(5)$$

$$\left[\frac{\partial^2 z}{\partial r \partial t}\right] = G\left(\frac{dA}{d\sigma} - \bar{A}\right) \qquad ...(6)$$

where $A = \left[\frac{\partial z}{\partial r}\right]$ and $\bar{A} = \left[\frac{\partial^2 z}{\partial r^2}\right]$ are the quantities defined over the discontinuity surface $S(t)$ and they may be regarded as functions of the distance σ along each normal trajectory. The quantity z may represent any of the slow-field variables and the square bracket [ ] stands for the value of the quantity enclosed immediately behind the wave surface minus its value just ahead of the wave surface. Since the medium ahead of the wave is uniform, the bracket expressions in the above relations are identical with the corresponding derivatives at the rear of the wave surface $S(t)$.

Speed of Propagation

Evaluating eqns. (1) - (4) on the inner boundary of $s(t)$ and using the relation (5) we get

$$(G - u)\,\zeta - \rho\lambda = 0 \qquad ...(7)$$

$$\xi - (G - u)\,\rho\lambda + \mu\, H\eta = 0 \qquad ...(8)$$

$$H\lambda - (G - u)\,\eta = 0 \qquad ...(9)$$

$$(G - u)\,\zeta - \rho\lambda C^2 = 0 \qquad ...(10)$$

where ξ, ζ, λ and η are respectively the values of the first order derivatives of p, ρ, u and H with respect to r, evaluated just at the rear of the discontinuity surface $s(t)$.

Equations (7) - (10) constitute a set of four homogeneous equations in four unknowns ξ, ζ, λ and η. Hence the necessary condition for non-trivial solutions of (7) - (10) is that

$$(G - u)^2 \{(G - u)^2 - C^2_{eff}\} = 0 \qquad ...(11)$$

where $C^2_{eff} = C^2 + b^2$; C and b are respectively the sound speed $(\gamma p/\rho)^{1/2}$ and the Alfven speed $(\mu H^2/\rho)^{1/2}$ just ahead of the wave surface, since the flow and field variables are continuous across $s(t)$.

Condition (11) suggests that either $G - u = \pm C_{eff}$ or $G - u = 0$. The later case in which the surface moves with the fluid will be discarded as uninteresting and we assume, without any loss of generality that

$$G - u = C_{eff} \qquad ...(12)$$

which determines the speed of propagation of the weak discontinuity surface $S(t)$. When the medium ahead of the wave is at rest, it follows from (12) that the wave surface $s(t)$ propagates with effective sound speed C_{eff}. We shall be interested in the rest of the paper in the propagation of $s(t)$ into a uniform medium at rest.

Equations (7) – (10) yield the following relations

$$\zeta = \xi/C^2 = \rho\lambda/C_{eff} = \rho\eta/H. \qquad ...(13)$$

Growth and Decay of Discontinuities

If we differentiate eqns. (1) – (4) with respect to r, evaluate on the inner boundary of $s(t)$ and make use of the relations (6) and (13) we get

$$\frac{d\lambda}{d\sigma} = \frac{1}{C_{eff}}\left\{\rho\bar{\lambda}\, C_{eff} - \bar{\xi} - \mu\, H\bar{\eta} - \frac{\mu H^2\lambda^2}{C^2_{eff}}\right\} \qquad ...(14)$$

$$\frac{d\zeta}{d\sigma} = \bar{\zeta} - \frac{\rho}{C_{eff}}\,\bar{\lambda} - \left(\frac{2}{\rho}\,\zeta + \frac{\alpha}{R}\right)\zeta \qquad ...(15)$$

$$\frac{d\eta}{d\sigma} = \bar{\eta} - \frac{H}{C_{eff}}\,\bar{\lambda} - \left(\frac{2}{H}\,\eta + \frac{\alpha}{R}\right)\eta \qquad ...(16)$$

$$\frac{d\xi}{d\sigma} = \bar{\xi} - \frac{\rho C^2}{C_{eff}}\,\bar{\lambda} - \left\{\frac{(1+\gamma)\,\xi}{\rho c^2} + \frac{\alpha}{R}\right\}\xi \qquad ...(17)$$

where $\bar{\lambda}$, $\bar{\xi}$, $\bar{\eta}$ and $\bar{\zeta}$ are respectively the values of second order derivatives of u, p, H and ρ with respect to r, evaluated just at the rear of the discontinuity surface $S(t)$.

Relations (13) yield

$$\frac{d\zeta}{d\sigma} = \frac{1}{c^2}\,\frac{d\zeta}{d\sigma} = \frac{\rho}{C_{eff}}\,\frac{d\lambda}{d\sigma} = \frac{\rho}{H}\,\frac{d\eta}{d\sigma}. \qquad ...(18)$$

Eliminating σ derivatives in eqns. (14) – (17) with the help of (18) and making use of (13) we get

$$\bar{\xi} - \frac{\rho c^2}{C_{eff}}\,\bar{\lambda} = \frac{(1+\gamma)\,\xi^2}{2\rho C^2_{eff}}\left\{1 + \left(\frac{2\gamma-1}{\gamma+1}\right)\frac{b^2}{c^2}\right\} + \frac{\alpha\xi}{2R} \qquad ...(19)$$

$$\bar{\zeta} - \frac{\rho}{C_{eff}}\,\bar{\lambda} = \frac{\zeta^2}{2\rho C^2_{eff}}\left\{b^2 + (3-\gamma)\,c^2\right\} + \frac{\alpha\zeta}{2R} \qquad ...(20)$$

$$\bar{\eta} - \frac{H}{C_{eff}}\,\bar{\lambda} = \frac{\eta^2}{2HC^2_{eff}}\left\{b^2 + (3-\gamma)\,c^2\right\} + \frac{\alpha\eta}{2R} \qquad ...(21)$$

$$\rho\bar{\lambda}\, C_{eff} - \bar{\xi} - \mu H\bar{\eta} = -\frac{\rho\lambda}{2}\left\{\frac{\lambda C^2}{C^2_{eff}}\left(1 + \gamma + \frac{b^2}{c^2}\right) + \frac{\alpha C_{eff}}{R}\right\}. \qquad ...(22)$$

Using (19) – (22) in (14) – (17), we get

$$\frac{d\lambda}{d\sigma} + \frac{\alpha\lambda}{2R} + \frac{\Lambda}{C_{eff}}\lambda^2 = 0 \quad \text{...(23)}$$

$$\frac{d\zeta}{d\sigma} + \frac{\alpha\zeta}{2R} + \frac{\Lambda}{\rho}\zeta^2 = 0 \quad \text{...(24)}$$

$$\frac{d\eta}{d\sigma} + \frac{\alpha\eta}{2R} + \frac{\Lambda}{H}\eta^2 = 0 \quad \text{...(25)}$$

$$\frac{d\xi}{d\sigma} + \frac{\alpha\xi}{2R} + \frac{\Lambda}{\rho c^2}\xi^2 = 0 \quad \text{...(26)}$$

where

$$\Lambda = (3b^2 + (\gamma + 1)\, c^2)/2C^2_{eff}.$$

Equations (23) – (26) are the transport equations for the quantities λ, ζ, η and ξ which we have been seeking. In view of the relations (13), eqns. (24) – (26) are derivable from (23) and therefore, eqn. (23) is sufficient to predict the growth or decay of the weak discontinuity associated with the wave surface $s(t)$. Equation (23) on integration yields

$$\lambda = \frac{\lambda_0(R_0/R)^{\alpha/2}}{1 + [\Lambda\lambda_0 R_0^{\alpha/2}\,(R^{(2-\alpha)/2} - R_0^{(2-\alpha)/2})/C_{eff}(1 - \frac{1}{2}\alpha)]} \quad \text{...(27)}$$

where λ_0 is the value of λ at the wave front at an instant t_0.

Equation (27) gives the variation of the discontinuity associated with $s(t)$ as it moves into a uniform gas at rest. We shall consider, in detail, the following two cases of plane and cylindrical wave fronts for which $\alpha = 0$ and 1 respectively.

Case I: *Plane waves*

For a plane motion (27) reduces to

$$\lambda = \frac{\lambda_0}{1 + [\Lambda\,\lambda_0(R - R_0)/C_{eff}]} \quad \text{...(28)}$$

which, in the limit of zero magnetic field corresponds exactly to that obtained by Thomas (1957b) and Pack (1960) for the gradient at the head of a plane wave in an ordinary gas. It follows from (28) that if $\lambda_0 > 0$ (i.e. an expansive wave front) then λ remains positive for all $R > R_0$ and approaches zero as $R \to \infty$. Thus the wave decays and damps out ultimately. On the other hand, if $\lambda_0 < 0$ (i.e. a compressive wave front) then λ increases beyond all bounds for a finite R approaching the value

$$R_c = R_0 + (C_{eff}/\Lambda \mid \lambda_0 \mid) \qquad ...(29)$$

i.e. at the instant

$$t_c = t_0 + (1/\Lambda \mid \lambda_0 \mid) \qquad ...(30)$$

the velocity gradient at the front becomes infinite, which signifies the appearance of a shock wave. An ordinary gasdynamic result corresponding to (30) is in the form

$$t_c = t_0 + \frac{2}{(\gamma + 1) \mid \lambda_0 \mid}. \qquad ...(31)$$

The comparison of (30) and (31) shows that for $\gamma < 2$, the magnetic field effects are to decrease the shock formation time t_c relative to what it would be in the absence of magnetic field.

Case II : *Cylindrical waves*

For a cylindrical wave (28) reduces to

$$\lambda = \frac{\lambda_0(R_0/R)^{1/2}}{1 + (2\Lambda\lambda_0 R_0(\sqrt{(R/R_0)} - 1)/C_{eff})}. \qquad ...(32)$$

This result in the limit of zero magnetic field corresponds exactly to that derived by Thomas (1957b) and Pack (1960) for the gradient at the head of a cylindrical wave. It follows from (32) that if $\lambda_0 > 0$ then λ remains positive for all $R > R_0$ and approaches zero as $R \to \infty$. But, if $\lambda_0 < 0$ then λ becomes infinite at a finite $R = R_c$ given b

$$R_c = R_0 \{1 + (C_{eff}/2\Lambda R_0 \mid \lambda_0 \mid)^2 \qquad ...(33)$$

and the wave terminates into a shock. Thus a cylindrical wave with a compression at its head steepens up into a shock at an instant

$$t_c = t_0 + (1/\Lambda \mid \lambda_0 \mid) \{1 + (C_{eff}/4\Lambda R_0 \mid \lambda_0 \mid)\} \qquad ...(34)$$

Comparing (34) with (30), we find that for a cylindrical motion, the time t_c taken for the shock formation is greater than the corresponding time t_c for a plane motion.

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