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FLOW INDUCED BY UNHARMONIC OSCILLATIONS OF A SOLID SPHERE IN A VISCOUS FLUID

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This paper deals with the flow induced by a solid sphere performing torsional unharmonic oscillations in an unbounded, incompressible viscous fluid. It is found that in the second approximation there exists a steady part of torque which is absent in the harmonic case.

1. INTRODUCTION

Carrier and Di Prima (1956) studied the torsional harmonic oscillations of a solid sphere in a viscous fluid taking into account the effect of centrifugal forces. Miyagi and Nakahasi (1975) studied the translational unharmonic oscillations of a circular cylinder in a viscous fluid and showed that the slow steady secondary flow occurs outside the boundary layer around the cylinder due to the asymmetry in the motion of circular cylinder.

In the present paper, we have discussed the motion of the fluid induced by the torsional unharmonic oscillations of a solid sphere in an unbounded, incompressible viscous fluid. Carrying out the first order approximation and then using it, the second order approximation has been computed which also gives a steady part of the motion. The velocity of the fluid in the vicinity of the sphere has been obtained for calculating the torque on the sphere for large non-dimensional parameter a. The results given by Carrier and Di Prima (1956) are obtained as a particular case.

2. EQUATION OF MOTION

We consider a sphere of radius R performing unharmonic torsional oscillations with angular velocity $\epsilon\omega(e^{i\omega t} + ke^{2i\omega t})$, ϵ being a small non-dimensional parameter governing the angular displacement in a viscous fluid of density ρ and kinematic viscosity ν. The motion is governed by

$$\text{div.}\, \vec{V} = 0 \qquad \ldots(2.1a)$$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} . \text{grad})\, \vec{V} = -\frac{1}{\rho}\, \text{grad}\, p + \nu \nabla^2 \vec{V}. \qquad \ldots(2.1b)$$

It will be convenient to express as in Carrier and Di Prima (1956) that

$$\vec{V} = (r \sin \theta)^{-1} (\psi \sin \theta)_\theta \, \hat{e}_r - r^{-1}(\psi r)_r \, \hat{e}_\theta + v\hat{e}_\phi \quad \text{...(2.2)}$$

where $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ are the unit vectors in the spherical polar coordinate system (r, θ, ϕ) with pole at the centre of the sphere and axis along the axis of rotation.

Introducing the non-dimensional quantities

$$\left.\begin{aligned} & T = \omega t,\ x = r/R;\ a^2 = \frac{R^2 \rho\omega}{2\mu} = \frac{R^2\omega}{2\nu} \\ \text{and} \quad & \\ & v = \epsilon\omega R\,[F_0(x, \theta, T) + \epsilon^2 F_1(x, \theta, T)] \\ & \psi = \epsilon^2 \omega R^2\, G_0(x, \theta, T). \end{aligned}\right\} \quad \text{...(2.3)}$$

We can deduce from eqns. (2.1) and (2.2), the following set of equations

$$2a^2(F_0)_T + L(F_0) = 0 \quad \text{...(2.4)}$$

$$2a^2(G_0)_T + L^2(G_0) = 2a^2x^{-1}\{\cot\theta(F_0^2)_x - x^{-1}(F_0^2)_\theta\} \quad \text{...(2.5)}$$

$$\begin{aligned} 2a^2(F_1)_T + L(F_1) = &- 2a^2(x^2 \sin\theta)^{-1}\{(xF_0)_x(G_0 \sin\theta)_\theta \\ &- (xG_0)_x(F_0 \sin\theta)_\theta\} \end{aligned} \quad \text{...(2.6)}$$

where

$$L(f) = -x^{-1}\left[(xf)_{xx} + x^{-1}\left\{\frac{(f \sin\theta)_\theta}{\sin\theta}\right\}_\theta\right]. \quad \text{...(2.7)}$$

The boundary conditions are the no-slip condition on the sphere and the vanishing of velocity at infinity. Thus, in terms of F_0, F_1 and G_0, we have

$$F_0(x, \theta, T) = (e^{iT} + ke^{2iT}) \sin\theta \quad \text{...(2.8)}$$

$$F_1 = G_0 = (G_0)_x = 0 \text{ at } x = 1 \quad \text{...(2.9)}$$

$$F_0 = G_0 = (G_0)_x = 0 \text{ as } x \to \infty\,. \quad \text{...(2.10)}$$

3. First Order Approximation

Boundary conditions suggest that we seek a solution of eqn. (2.4) of the form

$$F_0(x, \theta, T) = [F_{01}(x)\, e^{iT} + kF_{02}(x)\, e^{2iT}] \sin\theta. \quad \text{...(3.1)}$$

Equation (2.4) provides

$$L_1(F_{01}) - z^2F_{01} = 0 \quad \text{...(3.2)}$$

$$L_1(F_{02}) - 2z^2F_{02} = 0 \quad \text{...(3.3)}$$

where

$$\left.\begin{aligned} L_1(f) &= x^{-1}\{(xf)_{xx} - 2x^{-1}f\} \\ z &= (1+i)\,a. \end{aligned}\right\} \qquad \ldots(3.4)$$

These equations have to be solved under the boundary conditions

$$F_{01} = F_{02} = 1 \text{ at } x = 1;\ F_{01}, F_{02} \to 0 \text{ as } x \to \infty. \qquad \ldots(3.5)$$

The solution of eqn. (3.2) was obtained earlier by Carrier and Di Prima (1956). Now solving eqn. (3.3) for F_{02} and combining this with F_{01}, we have from eqn. (3.1)

$$F_0(x, \theta, T) = \left[\frac{(1+zx)}{(1+z)\,x^2}\, e^{-z(x-1)}\, e^{iT} + k\,\frac{(1+\sqrt{2}\,zx)}{(1+\sqrt{2}\,z)\,x^2} \exp(-\sqrt{2}\,z(x-1))\, e^{2iT}\right] \sin\theta \qquad \ldots(3.6)$$

$$= \left\{\frac{1 + 2ax + 2a^2x^2}{(1+2a+2a^2)\,x^4}\right\}^{1/2} e^{-a(x-1)} \times \sin\theta \exp(i(T - a(x-1) + \eta_1))$$
$$+ k\left\{\frac{(1 + 2\sqrt{2}\,ax + 4a^2x^2)}{(1 + 2\sqrt{2}\,a + 4a^2)\,x^4}\right\}^{1/2} \exp(-\sqrt{2}\,a(x-1)) \times \sin\theta \exp(i(2T - \sqrt{2}\,a(x-1) + \eta_2)) \qquad \ldots(3.7)$$

where

$$\eta_1 = \arc\tan \frac{a(x-1)}{(1+a+ax+2a^2x)},\quad \eta_2 = \arc\tan \frac{\sqrt{2}\,a(x-1)}{(1+\sqrt{2}\,a+\sqrt{2}\,ax+4a^2x)}.$$

The tangential stress opposing the motion of the sphere is given by $-\mu R(v/R)_r$ evaluated at $r = R$. The torque N_0 on the sphere is $-\mu R^2 \sin\theta (v/R)_r$ integrated over the sphere. The first order approximation to the torque on the sphere can be evaluated from expressions (3.7) for F_0 and is

$$N_0 = -2\pi\mu\epsilon\omega R^3 \int_0^\pi \frac{\partial}{\partial x}\left(\frac{F_0}{x}\right)_{x=1} \sin^2\theta\, d\theta$$

$$= -\frac{8}{3}\pi\mu\epsilon\omega R^3 \left[\frac{(8a^6 + 32a^5 + 64a^4 + 84a^3 + 72a^2 + 36a + 9)^{1/2}}{(1 + 2a + 2a^2)} \times \exp i(T + \eta_3)\right.$$
$$\left. + k\,\frac{(64a^6 + 128\sqrt{2}\,a^5 + 256a^4 + 168\sqrt{2}\,a^3 + 144a^2 + 36\sqrt{2}\,a + 9)^{1/2}}{(1 + 2\sqrt{2}\,a + 4a^2)} \times \exp i(2T + \eta_4)\right] \qquad \ldots(3.8)$$

where

$$\eta_3 = \arc \tan \frac{2a^2(1+a)}{(3+6a+6a^2+2a^3)}$$

$$\eta_4 = \arc \tan \frac{4a^2(1+\sqrt{2}\,a)}{(3+6\sqrt{2}\,a+12a^2+4\sqrt{2}\,a^3)}.$$

For large a, this reduces to

$$N_0 = -\frac{8}{3}\pi\mu\epsilon\omega R^3\sqrt{2}\,a\Big[\Big(1+\frac{1}{a}\Big)\exp\,(i(T+\eta_5)) + k\Big(\sqrt{2}+\frac{1}{a}\Big)\exp\,(i(2T+\eta_6))\Big] \quad ...(3.9)$$

where

$$\eta_5 = \arc \tan \Big(1-\frac{2}{a}\Big),\ \eta_6 = \arc \tan \Big(1-\frac{\sqrt{2}}{a}\Big).$$

4. Circulatory Motion

We can now express

$$(Rl.F_0)^2 = (F_{01}\cos T + kF_{02}\cos 2T)^2 \sin^2\theta$$
$$= \tfrac{1}{2}[(F_{01}^2 + k^2F_{02}^2) + F_{01}^2 e^{2iT} + k^2F_{02}^2 e^{4iT} + 2kF_{01}F_{02}(e^{iT}+e^{3iT})]\sin^2\theta. \quad ...(4.1)$$

This suggests that we should seek solution of (2.5) as

$$G_0(x,\theta,T) = [f_0(x) + f_1(x)\,e^{iT} + f_2(x)\,e^{2iT} + f_3(x)\,e^{3iT} + f_4(x)\,e^{4iT}]\sin 2\theta. \quad ...(4.2)$$

Substituting the above in (2.5), we get the following differential equations for f_0, f_1, f_2, f_3 and f_4

$$L_2^2(f_0) = -\frac{3a^2+6a^3x+6a^4x^2+2a^5x^3}{(1+2a+2a^2)\,x^6}\,e^{-2a(x-1)} - \frac{k^2(3a^2+6\sqrt{2}\,a^3x+12a^4x^2+4\sqrt{2}\,a^5x^3)}{(1+2\sqrt{2}\,a+4a^2)\,x^6}\exp\,(-2\sqrt{2}\,a(x-1)) \quad ...(4.3)$$

$$L_2^2(f_1) - z^2L_2(f_1) = \frac{-ka^2[6+(6\sqrt{2}+6)\,zx+(6\sqrt{2}+3)\,z^2x^2+(\sqrt{2}+2)\,z^3x^3]}{(1+z)(1+\sqrt{2}\,z)\,x^6} \times \exp\,(-(\sqrt{2}+1)\,z(x-1)) \quad ...(4.4)$$

$$L_2^2(f_2) - 2z^2L_2(f_2) = \frac{-a^2(3+6zx+4z^2x^2+z^3x^3)}{(1+z)^2\,x^6}\,e^{-2z(x-1)} \quad ...(4.5)$$

$$L_2^2(f_3) - 3z^2L_2(f_3) = \frac{-ka^2[6 + (6\sqrt{2} + 6)\, zx + (6\sqrt{2} + 3)\, z^2x^2 + (\sqrt{2} + 2)\, z^3x^3]}{(1 + z)(1 + \sqrt{2}\, z)\, x^6} \times \exp(-(\sqrt{2} + 1)\, z(x - 1)) \quad \ldots(4.6)$$

$$L_2^2(f_4) - 4z^2L_2(f_4) = \frac{-k^2a^2(3 + 6\sqrt{2}\, zx + 8z^2x^2 + 2\sqrt{2}\, z^3x^3)}{(1 + \sqrt{2}\, z)^2\, x^6} \exp(-2\sqrt{2}\, z(x - 1)) \quad \ldots(4.7)$$

where

$$L_2(f) = x^{-1}\{(xf)_{xx} - 6x^{-1}f\}. \quad \ldots(4.8)$$

These are to be solved under the boundary conditions

$$\left.\begin{aligned} f_0 = f_1 = f_2 = f_3 = f_4 = 0 \text{ at } x = 1 \\ f_0 = f_1 = f_2 = f_3 = f_4 = 0 \text{ at } x \to \infty. \end{aligned}\right\} \quad \ldots(4.9)$$

Solutions of eqns. (4.3) to (4.7) satisfying the relevant boundary conditions are as follows:

$$\begin{aligned} f_0 = {} & \frac{A_0}{x} + \frac{B_0}{x} - \frac{1}{20(1 + 2a + 2a^2)} \\ & \times \left\{\left(\frac{3a}{2x^3} + \frac{a^2}{2x^2} - \frac{a^3}{3x} + \frac{a^4}{3} - \frac{2a^5x}{3}\right) e^{-2a(x-1)} \right. \\ & \left. + \frac{4a^6x^2e^{2a}}{3} E(2ax) \right\} \\ & - \frac{k^2}{20(1 + 2\sqrt{2}\, a + 4a^2)} \left\{\left(\frac{3\sqrt{2}\, a}{4x^3} + \frac{a^2}{2x^2} - \frac{\sqrt{2}\, a^3}{3x} + \frac{2a^4}{3} \right.\right. \\ & \left.\left. - \frac{4\sqrt{2}\, a^5x}{3}\right) \exp(-2\sqrt{2}\, a(x-1)) + \frac{16}{3} a^6x^2 \exp(2\sqrt{2}\, a)\, E(2\sqrt{2}\, ax)\right\} \end{aligned} \quad \ldots(4.10)$$

$$\begin{aligned} f_1 = {} & \frac{A_1}{x^3} + \frac{(3 + 3zx + z^2x^2)}{x^3} e^{-z(x-1)} B_1 \\ & - \frac{ka^2}{2(1 + z)(1 + \sqrt{2}\, z)} \left\{\frac{[(5\sqrt{2} - 4) + (\sqrt{2} - 1)\, zx]}{2zx^3} \right. \\ & \times \exp(-(\sqrt{2} + 1)\, z(x - 1)) \\ & - \frac{(3 + 3zx + z^2x^2)}{2zx^3} E(\sqrt{2}\, zx) \exp(\sqrt{2}\, z - z(x - 1)) \\ & \left. + \frac{(3 - 3zx + z^2x^2)}{2zx^3} E((2 + \sqrt{2})\, zx) \exp[(2 + \sqrt{2})\, z + z(x - 1)]\right\} \end{aligned} \quad \ldots(4.11)$$

$$f_2 = \frac{A_2}{x^3} + \frac{(3 + 3\sqrt{2}\,zx + 2z^2x^2)}{x^3} \exp(-\sqrt{2}\,z(x-1))\,B_2$$
$$- \frac{a^2}{16(1+z)^2}\left\{\left(\frac{3+zx}{zx^3}\right)e^{-2z(x-1)}\right.$$
$$- \frac{(3 + 3\sqrt{2}\,zx + 2z^2x^2)}{2\sqrt{2}\,zx^3} E((2-\sqrt{2})\,zx) \exp[(2-\sqrt{2})\,z - z(x-1)]$$
$$\left.+ \frac{(3 - 3\sqrt{2}\,zx + 2z^2x^2)}{2\sqrt{2}\,zx^3} E((2+\sqrt{2})\,zx) \exp[(2+\sqrt{2})\,z - z(x-1)]\right\} \quad \ldots(4.12)$$

$$f_3 = \frac{A_3}{x^3} + \frac{(3 + 3\sqrt{3}\,zx + 3z^2x^2)}{x^3} \exp(-\sqrt{3}\,z(x-1))\,B_3$$
$$- \frac{ka^2}{6(1+z)(1+\sqrt{2}\,z)}\left\{\frac{(6\sqrt{2}-5) + \sqrt{2}\,zx}{2zx^3} \exp(-(\sqrt{2}+1)\,z(x-1))\right.$$
$$- \frac{(1 + \sqrt{3}\,zx + z^2x^2)}{\sqrt{3}\,zx^3} E((1+\sqrt{2}-\sqrt{3})\,zx)$$
$$\times \exp[(1+\sqrt{2}-\sqrt{3})\,z - \sqrt{3}\,z(x-1)]$$
$$+ \frac{(1 - \sqrt{3}\,zx + z^2x^2)}{\sqrt{3}\,zx^3} E((1+\sqrt{2}+\sqrt{3})\,zx)$$
$$\left.\times \exp[(1+\sqrt{2}+\sqrt{3})\,z + \sqrt{3}z(x-1)]\right\} \quad \ldots(4.13)$$

$$f_4 = \frac{A_4}{x^3} + \frac{(3 + 6zx + 4z^2x^2)}{x^3} \exp(-2z(x-1))\,B_4$$
$$- \frac{k^2a^2}{32(1+\sqrt{2}\,z)^2}\left\{\left(\frac{3\sqrt{2}+2zx}{zx^3}\right) \exp(-2\sqrt{2}\,z(x-1))\right.$$
$$- \frac{(3 + 6zx + 4z^2x^2)}{2zx^3} E((2\sqrt{2}-2)\,zx) \exp[(2\sqrt{2}-2)z - 2z(x-1)]$$
$$\left.+ \frac{(3 - 6zx + 4z^2x^2)}{2zx^3} E((2\sqrt{2}+2)\,zx) \exp[(2\sqrt{2}+2)\,z + 2z(x-1)]\right\} \quad \ldots(4.14)$$

where

$$E(x) = \int_x^\infty \frac{e^{-u}}{u}\,du \quad \ldots(4.15)$$

and

$$A_0 = -\frac{a^2}{16(1+2a+2a^2)}\left\{1 + \frac{2a}{3} - \frac{2a^2}{3} + \frac{4a^3}{3} - \frac{8a^4}{3} e^{2a}E(2a)\right\} -$$

(equation continued on p. 955)

$$- \frac{k^2a^2}{16(1 + 2\sqrt{2}\,a + 4a^2)} \left\{1 + \frac{2\sqrt{2}\,a}{3} - \frac{4a^2}{3} + \frac{8\sqrt{2}\,a^3}{3} - \frac{32a^4}{3} \exp(2\sqrt{2}\,a)\,E(2\sqrt{2}\,a)\right\} \quad ...(4.16)$$

$$B_0 = \frac{a}{40(1 + 2a + 2a^2)} \left\{3 + \frac{7a}{2} + a^2 - a^3 + 2a^4 - 4a^5e^{2a}E(2a)\right\}$$
$$- \frac{k^2a}{40(1 + 2\sqrt{2}\,a + 4a^2)} \left\{\frac{3\sqrt{2}}{2} + \frac{7a}{2} + \sqrt{2}\,a^2 - 2a^3 + 4\sqrt{2}\,a^4 - 16a^5 \exp(2\sqrt{2}a)\,E(2\sqrt{2}\,a)\right\} \quad ...(4.17)$$

$$A_1 = - \frac{ka^2}{4(1 + z)^2\,(1 + \sqrt{2}\,z)} \left\{(6\sqrt{2} - 9) + \frac{(5\sqrt{2} - 10)}{z} + (\sqrt{2} - 2)\,z - \frac{3}{z^2} + 2z^2E((2 + \sqrt{2})\,z) \exp((2 + \sqrt{2})\,z)\right\} \quad ...(4.18)$$

$$B_1 = \frac{ka^2}{4(1 + z)^2\,(1 + \sqrt{2}\,z)\,z} \left\{1 + \frac{1}{z} - (1 + z)\,E(\sqrt{2}\,z) \exp(\sqrt{2}\,z) + (1 - z)\,E((2 + \sqrt{2})\,z) \exp((2 + \sqrt{2})\,z)\right\} \quad ...(4.19)$$

$$A_2 = - \frac{a^2}{16(1 + z)^2\,(1 + \sqrt{2}\,z)} \left\{1 + \frac{3\sqrt{2}}{z} + \frac{3}{z^2} - (\sqrt{2} - 2)\,z - z^2E((2 + \sqrt{2})\,z)\right\} \quad ...(4.20)$$

$$B_2 = \frac{a^2}{16(1 + z)^2\,(1 + \sqrt{2}\,z)\,z} \left\{1 + \frac{1}{z} - \frac{(1 + \sqrt{2}\,z)}{2\sqrt{2}} E((2 - \sqrt{2})\,z) \times \exp((2 - \sqrt{2})\,z) + \frac{(1 - \sqrt{2}\,z)}{2\sqrt{2}} E((2 + \sqrt{2})\,z) \exp((2 + \sqrt{2})\,z)\right\} \quad ...(4.21)$$

$$A_3 = - \frac{ka^2}{6(1 + z)\,(1 + \sqrt{2}\,z)\,(1 + \sqrt{3}\,z)} \left\{(12 + 6\sqrt{2} - 5\sqrt{6}) + \frac{(7\sqrt{6} + 12\sqrt{3} - 5\sqrt{2})}{z} + \frac{(12 + 7\sqrt{2})}{z^2} + (2 - \sqrt{2} - \sqrt{6})\,z - 2z^2E((1 + \sqrt{2} + \sqrt{3})\,z) \exp((1 + \sqrt{2} + \sqrt{3})\,z)\right\} \quad ...(4.22)$$

$$B_3 = \frac{ka^2}{18(1 + z)\,(1 + \sqrt{2}\,z)\,(1 + \sqrt{3}\,z)\,z} \left\{(2 + \sqrt{2}) + \frac{(12 + 7\sqrt{2})}{z} - \right.$$

(equation continued on p. 956)

$$- \frac{(1+\sqrt{3}\,z)}{\sqrt{3}} E((1+\sqrt{2}-\sqrt{3})\,z) \exp((1+\sqrt{2}-\sqrt{3})\,z)$$

$$+ \frac{(1-\sqrt{3}\,z)}{\sqrt{3}} E((1+\sqrt{2}+\sqrt{3})\,z) \exp((1+\sqrt{2}+\sqrt{3})\,z)\Big\} \quad ...(4.23)$$

$$A_4 = - \frac{k^2 a^2}{16(1+\sqrt{2}\,z)^2\,(1+2z)} \Big\{ 5 - \frac{(3\sqrt{2}-6)}{z} + \frac{3}{z^2}$$

$$- (2\sqrt{2}-2)\,z - 4z^2 E((2\sqrt{2}+2)\,z) \exp((2\sqrt{2}+2)\,z)\Big\} \quad ...(4.24)$$

$$B_4 = \frac{k^2 a^2}{32(1+\sqrt{2}\,z)^2\,(1+2z)\,z} \Big\{ -\sqrt{2} + \frac{2}{z} - \frac{(1+2z)}{2} E((2\sqrt{2}-2)z)$$

$$\times \exp((2\sqrt{2}-2)\,z) + \frac{(1-2z)}{2} E((2\sqrt{2}+2)\,z) \exp((2\sqrt{2}+2)\,z)\Big\}. \quad ...(4.25)$$

As the sphere performs aharmonic oscillations, the fluid sucked in along the axis of oscillation is thrown away along the equatorial plane of the sphere. The steady part of velocity in θ-direction is given by

$$v = - \frac{\epsilon\omega R \sin 2\theta}{x} \frac{d}{dx}(xf_0). \quad ...(4.26)$$

Since $\frac{d}{dx}(xf_0)$ vanishes at $x = 1$ and $x = \infty$ only, v is one signed. The amount Q of the fluid which flows across the surface of a cone of apex angle θ is the measure of the fluid sucked in along the axis of oscillation. It is given by

$$Q = \int_0^\infty 2\pi r \sin\theta v \, dr$$

$$= \frac{\pi\epsilon^2\omega R^3 a^2 \sin\theta \sin 2\theta}{8} \Big[\frac{3+2a-2a^2+4a^3-8a^4 e^{2a} E(2a)}{3(1+2a+2a^2)}$$

$$+ k^2 \frac{(3+2\sqrt{2}\,a-4a^2+8\sqrt{2}\,a^3) - 32a^4 \exp(2\sqrt{2}\,a) E(2\sqrt{2}\,a)}{3(1+2\sqrt{2}\,a+4a^2)} \Big] \quad ...(4.27)$$

We obtain the approximation for large a

$$Q = \frac{\pi\epsilon^2\omega R^3 \sin\theta \sin 2\theta}{16} \Big[\Big(1+\frac{k^2}{2}\Big) - \frac{(2+(k^2\,2^{-1/2}))}{a} + O(1/a^2) \Big]. \quad ...(4.28)$$

5. Second Order Approximation

Substituting for F_0 and G_0 in eqn. (2.6), we obtain

$$2a^2(F_1)_T + L(F_1) = [H_{01}R_1(\theta) + H_{03}R_3(\theta)] + [H_{11}R_1(\theta) + H_{13}R_3(\theta)]\, e^{iT} + \ldots\ldots + \ldots + + [H_{61}R_1(\theta) + H_{63}R_3(\theta)]\, e^{6iT} \quad \ldots(5.1)$$

where $R_n(\theta)$ represents the derivative of Legendre's polynomials P_n (cos θ) and

$$H_{01} = \frac{4a^2}{5x^2} [(xf_1)_x F_{01} + (xF_{01})_x f_1 + k(xf_2)_x F_{02} + k(xF_{02})_x f_2] \quad \ldots(5.2)$$

$$H_{03} = \frac{8a^2}{5x^2} [2(xf_1)_x F_{01} - 3(xF_{01})_x f_1 + 2k(xf_2)_x F_{02} - 3k(xF_{02})_x f_2] \quad \ldots(5.3)$$

$$H_{11} = \frac{4a^2}{5x^2} [2(xf_0)_x F_{01} + 2(xF_{01})_x f_0 + k(xf_1)_x F_{02} + k(xF_{02})_x f_1 + (xf_2)_x F_{01} + (xF_{01})_x f_2 + k(xf_3)_x F_{02} + k(xF_{02})_x f_3] \quad \ldots(5.4)$$

$$H_{13} = \frac{8a^2}{5x^2} [4(xf_0)_x F_{01} - 6(xF_{01})_x f_0 + 2k(xf_1)_x F_{02} - 3k(xF_{02})_x f_1 + 2(xf_2)_x F_{01} - 3(xF_{01})_x f_2 + 2k(xf_3)_x F_{02} - 3k(xF_{02})_x f_2]. \quad \ldots(5.5)$$

Equation (5.1) suggests that we should seek a solution of F_1 of the form

$$F_1(x, \theta, T) = [p_{01}R_1(\theta) + p_{03}R_3(\theta)] + [p_{11}R_1(\theta) + p_{13}R_3(\theta)]\, e^{iT} + \ldots . \quad \ldots(5.6)$$

Substituting in (5.1), we get

$$(xp_{01})_{xx} - 2x^{-1}p_{01} = -xH_{01} \quad \ldots(5.7)$$

$$(xp_{03})_{xx} - 12x^{-1}p_{03} = -xH_{03} \quad \ldots(5.8)$$

and similar equations for other unknowns.

It is interesting to note that F_1 involves a steady part also which gives rise to a steady torque and we turn our attention to its determination through the solution of eqns. (5.7) and (5.8). The functions H_{01} and H_{03} occurring in (5.7) and (5.8) are extremely complicated functions involving several products of different exponentials and polynomials in $1/x$ as well as exponential integrals. To get the exact solutions, as such, of these equations is a laborious task. Therefore as in Carrier and Di Prima (1956), we shall confine ourselves to large a case. For calculating the torque, the velocities in the vicinity of the sphere are needed. Therefore, we take $x = 1 + \xi/a$ and neglecting the terms of $(1/a^2)$, eqns. (5.7) and (5.8) reduce respectively to the form

$$\frac{d^2}{d\xi^2}(xp_{01}) = k\left(l_j + \frac{m_j}{a} + \frac{n_j\xi}{a}\right) \sum_{j=1}^{5} \exp\left(-\frac{\delta_j\xi}{a}\right) \quad \ldots(5.9)$$

and

$$\frac{d^2}{d\xi^2}(xp_{03}) = k\left(l'_j + \frac{m'_j}{a} + \frac{n'_j\xi}{a}\right)\sum_{j=1}^{5} \exp\left(-\frac{\delta_j\xi}{a}\right), \qquad ...(5.10)$$

where l_j, m_j, n_j, l'_j, m'_j and n'_j are known complex quantities independent of a and ξ. The δ_j are z, $\sqrt{2}\,z$, $2z$, $2\sqrt{2}\,z$ and $(2 + \sqrt{2})\,z$ respectively.

Integrating the above equation and applying the relevent boundary conditions, we obtain

$$p_{01} = k\left(c_j + \frac{d_j}{a} + \frac{e_j\xi}{a}\right)\left[1 + \sum_{j=1}^{5} \exp\left(-\frac{\delta_j\xi}{a}\right)\right] \qquad ...(5.11)$$

$$p_{03} = k\left(c'_j + \frac{d'_j}{a} + \frac{e'_j\xi}{a}\right)\left[1 + \sum_{j=1}^{5} \exp\left(-\frac{\delta_j\xi}{a}\right)\right]. \qquad ...(5.12)$$

where c_j, c'_j, e_j and e'_j are known real and d_j, d'_j are known complex quantities independent of a and ξ.

However, in calculating the torque N_1, p_{03} does not contribute to it because of the orthogonality relation in the Legendre's polynomials. Thus evaluating

$$\frac{d}{dx}(p_{01}/x) \text{ at } x = 1$$

we obtain

$$N_1 = -2\pi\epsilon\mu\omega R^3 \int_0^\pi \frac{\partial}{\partial x}\left(\frac{p_{01}}{x}\right)_{x=1} \sin^2\theta\, d\theta$$

$$= -(5.688)\,\pi\epsilon\mu\omega k R^3. \qquad ...(5.13)$$

The expression (5.13) shows that N_1 linearly increases as k increases. Letting $k = 0$ it should be noticed that N_1 vanishes and for eqn. (3.9), we recover the value of N_0 for harmonic oscillations as given in Carrier and Di Prima (1956).

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