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EXTENSION OF SIMPLEX TECHNIQUE FOR SOLVING FRACTIONAL PROGRAMMING PROBLEM

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The present paper describes the method for solving the linear fractional functional programming problem by the simplex method in which three basic variables are replaced by three non-basic variables at a time which is itself the generalization of the classical method. It reduces the time of computation for solving the above problem.

1. INTRODUCTION

Charnes and Cooper (1962) solved a programming problem with linear fractional functionals by resolving it into two linear programming problems, and Swarup (1965) developed a simplex technique for the same problem. The mathematical model of the problem is as follows :

$$\text{Maximize } Z = \frac{c^1 x + \alpha}{d^1 x + \beta} \qquad \ldots(1.1)$$

subject to the constraint $Ax = b, x \geqslant 0$

where x, c and d are $n \times 1$ vectors, b is an $m \times 1$ vector, c^1, d^1 denote transpose of vectors c and d is an $m \times n$ matrix, and α, β are scalar constants.

It is assumed that the constraint set

$$S = \{x : Ax = b, x \geqslant 0\}$$

is non-empty and bounded.

Here, our aim is to investigate the simplex technique for solving the linear fractional functional programming problem replacing three basic variables by three non-basic variables at a time which in fact reduces the time of computation of the problem. We elucidate our technique through a numerical example in section 2, which gives details of various steps. In section 3 we present the algorithm.

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2. Numerical Example

$$\text{Maximize} = \frac{x_1 + 2x_2 + 4x_3 + 5x_4 + 8x_5}{2x_1 + 5x_2 + 3x_3 + 4x_4 + 6x_5 + 1}$$

subject to
$$2x_1 + x_2 + 3x_3 + x_4 + x_5 \leqslant 15$$
$$x_1 + 2x_2 + 2x_5 \leqslant 8$$
$$3x_1 + 5x_2 + 2x_4 \leqslant 10$$
$$2x_2 + 4x_3 + 3x_4 + x_5 \leqslant 21$$
$$x_1, x_2, x_3, x_4, x_5 \geqslant 0.$$

After adding slack variables x_6, x_7, x_8 and x_9, the constraints become

$$2x_1 + x_2 + 3x_3 + x_4 + x_5 + x_6 = 15$$
$$x_1 + 2x_2 + 2x_5 + x_7 = 8$$
$$3x_1 + 5x_2 + 2x_4 + x_8 = 10$$
$$2x_2 + 4x_3 + 3x_4 + x_5 + x_9 = 21$$
$$x_j \geqslant 0,\ j = 1, ..., 9.$$

As in the simplex technique for a linear programming problem, our initial basic feasible solution is given by considering the slack variables as the basic variables (Hadley 1972, Ghosal 1979).

The initial basic feasible solution is given in Table I.

Table I

		c_j	1	2	4	5	8						Min. ratio
		d_j	2	5	3	4	6						x_{bi}/y_{ij}
d_B	c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b	
0	0	x_6	2	1	3	1	1	1	0	0	0	15	15/3 (2)
0	0	x_7	1	2	0	0	2	0	1	0	0	8	8/2 (1)
0	0	x_8	3	5	0	0	0	0	0	1	0	10	
0	0	x_9	0	2	4	3	1	0	0	0	1	21	21/3 (3)
$Z_1 = 0$ $Z_2 = 1$		$Z=0$											
$c_j - z_j^{(1)}$			1	2	4	5	8	—	—	—	—		
$d_j - z_j^{(2)}$			2	5	3	4	6	—	—	—	—		
Δ_j			1	2	4	5	8	—	—	—	—		

where $$Z_1 = c_B^{(1)} x_B + \alpha = (0, 0, 0, 0) \begin{pmatrix} 15 \\ 8 \\ 10 \\ 21 \end{pmatrix} + 0 = 0$$

$$Z_2 = d_B^{(1)} x_B + \beta = (0, 0, 0, 0) \begin{pmatrix} 15 \\ 8 \\ 10 \\ 21 \end{pmatrix} + 1 = 1$$

$$\triangle_j = Z_2(c_j - z_j^{(1)}) - Z_1(d_j - z_j^{(2)}) \text{ as derived in Swarup (1965).}$$

Since $Z_1 = 0$ and $Z_2 = 1$, therefore $\triangle_j = c_j - z_j^{(1)}$ as shown in Table I. The most convenient and time saving device for choosing the vector a_k in the basis B is

$$c_k - z_k = \max_j (c_j - z_j) \text{ for } c_j - z_j > 0.$$

Here, we choose three maximum $c_k - z_k$ at a time. Thus x_3, x_4 and x_5 are brought to the basis in the second step.

We select $b_1(= x_6)$, $b_2(= x_7)$ and $b_3(= x_9)$, the column vectors of the basis matrix for replacement by column vectors $a_3(= x_3)$, $a_4(= x_4)$ and $a_5(= x_5)$ by ranking ratios (x_{Bi}/y_{ij}) or b/a in ascending order, i.e. the min. 2nd row is replaced by x_5; the first row by x_3, etc., where minimum ratio rule is

$$\frac{x_{Br}}{y_{rj}} = \min_j \{x_{Bi}/y_{ij} : y_{ij} > 0\}$$

where $j = 1, 2, 3, 4, 5;\ r = 3, 4, 5.$

The new basis matrix is

$$\hat{B} = (\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4) = (a_3, a_4, \hat{b}_3, a_5)$$

$$= \begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix}$$

In this way we obtain a new basic feasible solution from the original in which b_1, b_2 and b_4 are replaced by three vectors a_3, a_5 and a_4 in A but not in B respectively as shown in Table II.

TABLE II

		c_j	1	2	4	5	8					
		d_j	2	5	3	4	6					
d_B	C_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b
3	4	x_3	1	$-1/5$	1	0	0	$3/5$	$-1/5$	0	$+1/5$	$16/5$
6	8	x_5	$1/2$	1	0	0	1	0	$1/2$	0	0	4
0	0	x_8	6	$19/5$	0	0	0	$8/5$	$-1/5$	1	$-6/5$	$36/5$
4	5	x_4	$-3/2$	$3/5$	0	1	0	$-4/5$	$1/10$	0	$3/5$	$7/5$
$Z_1 = 259/5$ $Z_2 = 201/5$		$Z = 259/201$										
		$c_j - z_j^{(1)}$	$1/2$	$-41/5$	—	—	—	$8/5$	$-37/10$	—	$-19/5$	
		$d_j - z_j^{(1)}$	2	$-14/5$	—	—	—	$7/5$	$-14/5$	—	$-9/5$	
		Δ_j	$-109/2$	$-1561/5$	—	—	—	$133/5$	$-791/10$	—	$-3112/25$	

$$Z_1 = c_B^2 \hat{x}_B + \alpha = (4, 8, 0, 5)\begin{pmatrix} 16/5 \\ 4 \\ 36/5 \\ 7/5 \end{pmatrix} + 0 = 4 \times \frac{16}{5} + 8 \times 4 + 0 \times \frac{36}{5} + 5 \times \frac{7}{5} + 0 = \frac{259}{5}$$

$$Z_2 = d_B^2 \hat{x}_B + \beta = (3, 6, 0, 4)\begin{pmatrix} 16/5 \\ 4 \\ 36/5 \\ 7/5 \end{pmatrix} + 1 = 3 \times \frac{16}{5} + 6 \times 4 + 0 \times \frac{36}{5} + 4 \times \frac{7}{5} + 1 = \frac{201}{5}$$

$$Z = \frac{Z_1}{Z_2} = \frac{259/5}{201/5} = \frac{259}{201}$$

$$\Delta_1 = Z_1(c_1 - z_1^{(1)}) - Z_2(d_1 - z_1^{(2)}) = \frac{259}{5} \times \frac{1}{2} - \frac{201}{5} \times 2 = -\frac{109}{2}$$

$$\triangle_2 = Z_1(c_2 - z_2^{(1)}) - Z_2(d_2 - z_2^{(2)}) = \frac{259}{5} \times -\frac{41}{5} - \frac{201}{5} \times -\frac{14}{5}$$
$$= -\frac{1561}{5}$$

$$\triangle_3 = \triangle_4 = \triangle_5 = 0$$

$$\triangle_6 = Z_1(c_6 - z_6^{(1)}) - Z_2(d_6 - z_6^{(2)}) = \frac{259}{5} \times \frac{8}{5} - \frac{201}{5} \times \frac{7}{5}$$
$$= +\frac{133}{5}$$

$$\triangle_7 = Z_1(c_7 - z_7^{(1)}) - Z_2(d_7 - z_7^{(2)}) = \frac{259}{5} \times -\frac{37}{10} - \frac{201}{5} \times -\frac{14}{5}$$
$$= -\frac{791}{10}$$

$$\triangle_8 = 0$$

$$\triangle_9 = Z_1(c_8 - z_8^{(1)}) - Z_2(d_8 - z_8^{(2)}) = \frac{259}{5} \times -\frac{19}{5} - \frac{201}{5} \times -\frac{9}{5}$$
$$= -\frac{3112}{25}.$$

Since each a_3, a_4 and a_5 can be expressed as the linear combination of vectors in $\hat{B}$, i.e.

$$a_3 = b_1 Y_{13} + b_2 y_{23} + b_4 Y_{43} + \sum_{i \neq 1, 2, 4} b_i y_{i3}$$

$$a_4 = b_1 y_{14} + b_2 Y_{24} + b_4 Y_{44} + \sum_{i \neq 1, 2, 4} b_i y_{i4}$$

and

$$a_5 = b_1 y_{15} + b_2 y_{25} + b_4 y_{45} + \sum_{i \neq 1, 2, 4} b_i y_{i5}.$$

Solving these equations for b_1, b_2 and b_4 by the formula

$$b_j = k_j/k;\ j = 1, 2, 4,\ k \neq 0$$

where

$$k = \begin{vmatrix} y_{13} & y_{14} & y_{15} \\ y_{23} & y_{24} & y_{25} \\ y_{43} & y_{44} & y_{45} \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 4 & 3 & 1 \end{vmatrix} = -10 \neq 0$$

and k_j is the determinant of coefficients of the variable y_{ij}'s having first two columns of k for the value of b_i's respectively, e.g.

$$b_1 = -1/k \begin{vmatrix} y_{23} \; y_{43} & \sum\limits_{i \neq 1,2,4} b_i y_{i3} & -a_3 \\ y_{24} \; y_{44} & \sum\limits_{i \neq 1,2,4} b_i y_{i4} & -a_4 \\ y_{25} \; y_{45} & \sum\limits_{i \neq 1,2,4} b_i y_{i5} & -a_5 \end{vmatrix}.$$

The new value of the objective function is given by

$$\hat{Z} = (\hat{c}_B \hat{x}_B + \alpha)/(\hat{d}_B \hat{x}_B + \beta) = \hat{Z}_1/\hat{Z}_2.$$

Here $\hat{Z}_1 = 259/5$, $\hat{Z}_2 = 201/5$. Thus $\hat{Z} = \hat{Z}_1/\hat{Z}_2 = 259/201$. Since all $\Delta_j < 0$ at the second stage, therefore the current solution is the optimal basic feasible solution.

3. ALGORITHM

Consider the linear fractional functional programming problem

$$\text{Maximize } Z = \frac{c^1 x + \alpha}{d^1 x + \beta} \qquad \ldots(3.1)$$

subject to the constraint $Ax = b$, $x \geqslant 0$

with the assumption that the denominator is positive for all feasible solutions.

Let x_B be the initial basic feasible solution of the given problem such that

$$Bx_B = b$$

$$x_B = bB^{-1} \qquad \ldots(3.2)$$

where $B = (b_1, b_2, \ldots, b_r, b_s, b_t, \ldots, b_m)$.

Further suppose that

$$Z_1 = c_B^1 x_B + \alpha$$

$$Z_2 = d_B^1 x_B + \beta$$

where c_B^1 and d_B^1 are the vectors having their components as the coefficients associated with the basic variables in the numerator and the denominator of the objective function respectively. In addition, we assume that for this basic feasible solution

$$Y_u = B^{-1}a_u, \quad y_v = B^{-1}a_v, \quad y_w = B^{-1}a_w;$$

$$Z_u^{(1)} = c_B^1 \, y_u, \quad Z_u^{(2)} = d_B^1 \, y_u;$$

$$Z_v^{(1)} = c_B^{(1)} \, y_v, \quad Z_v^{(2)} = d_B^1 \, y_v;$$

$$Z_w^{(1)} = c_B^1 \, y_w, \quad Z_w^{(2)} = d_B^1 \, y_w$$

are known for every column a_j of A not in B.

Now, we shall try to find out another basic feasible solution with improved value of $Z = Z_1/Z_2$ and confine our attention to those basic feasible solutions in which three columns of B are changed. Suppose the new basic feasible solution is denoted by $\hat{x}_B$, then

$$\hat{x}_B = \hat{B}^{-1}\hat{b}$$

where $B = (b_1, b_2, \ldots, a_u, a_r, a_w, \ldots, b_m)$ is another basis in which b_r, b_s and b_t are replaced by a_u, a_v, a_w respectively. The columns of the new matrix B are given by

$$\left.\begin{aligned} \hat{b}_i &= b_i \\ \hat{b}_r &= a_u \\ \hat{b}_s &= a_v \\ \hat{b}_t &= a_w \end{aligned}\right\} \qquad \ldots(3.3)$$

We obtain the values of the new basic variables in terms of the original ones, and y_{iu}, y_{iv} and y_{iw}.

The condition for the selection of uth, vth, wth columns of A not in B in case of maximization problem is : To select the uth, vth, wth columns of A for which $\triangle_u$, $\triangle_v$ and $\triangle_w$ are greatest positive. This has been illustrated through the numerical example (Section 2).

Also if degeneracy is not present, $\hat{Z} > Z$.

Thus, we can move from one basis to another by changing three vectors at a time so long as there is some a_j in A not in B, and at each step objective function is improved. This process will terminate when all $\triangle_j \leqslant 0$.

4. Remarks

1. When two non-basic variables happen to replace the same basic variables, i.e.

$$\min_i \, (x_{Bi} y_{iu} : y_{iu} > 0)$$

and

$$\min_i (x_{Bi} y_{iv} : y_{iv} > 0)$$

occur for the same value of i, then this difficulty can be overcome by choosing v'th column of matrix A (not in B) by alternative criterion such as choose the column of A the v'th for which $\triangle_{v'}$ is the greatest positive (Pranjipe 1965, Agarwal 1977, Kanchan 1976).

$$\triangle_j : j = 1, 2, 3, \ldots, n : j \neq u, v.$$

2. All the elements in the expression K are to be nonnegative. This difficulty can also be overcome by the same procedure as in Charnes and Cooper (1962).

5. Conclusion

The method presented here will reduce the number of iterations for a problem with larger dimensions.

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