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ALTERNATIVE APPROACH TO THE SIMPLEX METHOD

T. M. KARADE AND P. G. KUMBHARE

Department of Mathematics, Nagpur University, Nagpur 440010

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In this note an alternative approach to the simplex method of linear programming is suggested. The method sometimes involves less iterations than in the simplex method or at the most an equal number. This powerful technique is better understood by resolving a cycling problem.

INTRODUCTION

The linear programming has its own importance in obtaining the solution of a problem where two or more activities compete for limited resources. Mathematically we have to

$$\left.\begin{array}{r} \text{minimise* the objective function } CX \\ \text{subject to } AX = B \\ X \geqslant 0 \end{array}\right\} \quad \ldots(1)$$

where $X = n \times 1$ column vector,

$A = m \times n$ coefficient matrix,

$B = m \times 1$ column vector,

$C = 1 \times n$ row vector,

and the columns of A are denoted by $P_1, P_2, \ldots, P_n$.

There are four methods to obtain the solution of the above problem. These methods can be classified as:

(1) the graphical method,

(2) the systematic trial and error method,

(3) the vector method,

(4) the simplex method.

*Optimisation of an objective function can always be reduced to a minimisation of the objective function.

The simplex method is the most general and powerful. We now give a brief account of the simplex method (see Gass 1964). Consider a non-degenerate basic feasible solution

$$X_0 = (x_{10}, x_{20}, \ldots, x_{m0}, 0, \ldots, 0).$$

The corresponding value of the objective function is

$$x_{10}c_1 + x_{20}c_2 + \ldots + x_{m0}c_m = Z_0 \text{ (say).} \qquad \ldots(2)$$

It follows from the study of linear programming that for any fixed j, a set of feasible solutions can be constructed such that $Z < Z_0$ for any member of the set where

$$Z_j - c_j > 0.$$

The condition imposed on θ is $\theta = \min\limits_i \dfrac{x_{i0}}{x_{ij}} > 0$, $x_{ij} > 0$ for fixed j.

Dantzig's (1951) suggestion is to choose that entering vector corresponding to which $Z_j - c_j$ is maximum positive. It is shown that if we choose the vector P_j such that $\theta_j(Z_j - c_j)$ is maximum positive then the iterations required are fewer in some problems. This has been illustrated by giving the solution of a problem. We also show that either the iterations required are the same or less but iterations required are never more than those of the simplex method.

In what follows we shall illustrate the problem where the iterations are less (our method) than the simplex method.

$$\text{Maximize } Z = 4x_1 + 3x_2$$

$$\text{subject to } x_j \geqslant 0,\ j = 1, 2$$

and

$$\begin{aligned} x_1 + x_2 &\leqslant 50, \\ x_1 + 2x_2 &\geqslant 80, \\ 3x_1 + 2x_2 &\geqslant 40. \end{aligned}$$

The above problem can be framed as:

$$\text{Minimize } -4x_1 - 3x_2 + 0(x_3 + x_4 + x_5) + M(x_6 + x_7)$$

$$\text{subject to } x_j \geqslant 0,\ j = 1, 2, \ldots 7$$

and

$$\begin{aligned} x_1 + x_2 + x_3 &= 50 \\ x_1 + 2x_2 - x_4 + x_6 &= 80 \\ 3x_1 + 2x_2 - x_5 + x_7 &= 40 \end{aligned}$$

Initial Step

Basis	c_j	-4 P_1	-3 P_2	0 P_3	0 P_4	0 P_5	M P_6	M P_7	Solution
P_3	0	1	1	1	0	0	0	0	50
P_6	M	1	2	0	-1	0	1	0	80
P_7	M	3	2	0	0	-1	0	1	40
$Z_j - c_j$		$4M + 4$	$4M + 3$	0	$-M$	$-M$	0	0	

Here we have $Z_1 - c_1 = 4M + 4$ and $Z_2 - c_2 = 4M + 3$.

Corresponding minimum values of θ are $\theta_1 = 40/3$ and $\theta_2 = 20$. Therefore

$$\theta_1(Z_1 - c_1) = \frac{160}{3} M + \frac{160}{3}$$

$$\theta_2(Z_2 - c_2) = 80M + 60.$$

Since the value $80M + 60$ is maximum, we make P_2 as the entering vector in the basis.

Second Step

P_3	0	$-\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	30
P_6	M	-2	0	0	-1	1	1	-1	40
P_2	-3	$\frac{3}{2}$	1	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	20
$Z_j - c_j$		$-2M - \frac{1}{2}$	0	0	$-M$	$M + \frac{3}{2}$	0	$-2M - \frac{3}{2}$	

The entering vector is P_5.

Third Step

P_3	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	10
P_5	0	-2	0	0	-1	1	1	-1	40
P_2	-3	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	40
$Z_j - c_j$		$\frac{5}{2}$	0	0	$\frac{3}{2}$	0	$-M - \frac{3}{2}$	$-M$	

At this step $\theta_1(Z_1 - c_1) = 50$ and $\theta_4(Z_4 - c_4) = 30$.

Hence entering vector is P_1.

Fourth Step

P_1	-4	1	0	2	1	0	-1	0	20
P_5	0	0	0	4	1	1	-1	-1	80
P_2	-3	0	1	-1	-1	0	1	0	30
$Z_j - c_j$		0	0	-5	-1	0	$1-M$	$-M$	

If we solve this problem by the simplex method the above optimum solution is obtained at the fifth step. Thus the number of iterations required is reduced by our methodology.

It is known that the conventional simplex method is rather inconvenient in handling the degeneracy and cycling problems because here the choice of the vectors, entering and outgoing, plays an important role. The degeneracy occurs when there is a tie for outgoing vector. The possibility of cycling is crucial only if the current basic feasible solution has more than one variable zero. In the simplex method when there arises a tie for entering the vector, the vector with the lowest index j is selected. In our method the problem of tie in most of the degeneracy problems is solved. The powerfulness of our method lies in getting rid of the tie in the degeneracy problems. In such situations our technique positively is more powerful and handy as well. We shall illustrate this by considering the problem of Beale (1955) discussed in Gass (1964).

Problem

$$\text{Minimize } -\tfrac{3}{4}x_1 + 150x_2 - (1/50)\,x_3 + 6x_4$$

$$\text{subject to } \tfrac{1}{4}x_1 - 60x_2 - (1/25)\,x_3 + 9x_4 + x_5 = 0$$

$$\tfrac{1}{2}x_1 - 90x_2 - (1/50)\,x_3 + 3x_4 + x_6 = 0$$

$$x_3 + x_7 = 1$$

and $$x_i \geqslant 0,\ i = 1, 2, \ldots, 7.$$

Employing the simplex method the problem gives the identical solution at the seventh iteration and nothing is gained. Therefore the conventional simplex method is inconvenient as it does not provide proper guidance for selecting the entering and outgoing vectors.

We now employ the above method where we choose the entering vector for which $\theta_j(Z_j - c_j)$ is maximum positive where

$$\theta_j = \min_i\,(x_{i0}/x_{ij}),\quad x_{ij} > 0.$$

The optimum solution is then obtained at the third iteration.

Step I

Basis	C	$-\frac{3}{4}$ P_1	150 P_2	$-1/50$ P_3	6 P_4	0 P_5	0 P_6	0 P_7	Solution
P_5	0	$\frac{3}{4}$	-60	$-1/25$	9	1	0	0	0
P_6	0	$\frac{1}{2}$	-90	$-1/50$	3	0	1	0	0
P_7	0	0	0	1	0	0	0	1	1
		$\frac{3}{4}$	-150	$1/50$	-6	0	0	0	0

In this $\theta_1 = 0,\ Z_1 - c_1 = \frac{3}{4},\ \theta_1(Z_1 - c_1) = 0$

$\theta_3 = 1,\ Z_3 - c_3 = 1/50,\ \theta_3(Z_3 - c_3) = 1/50.$

Therefore the entering vector is P_3

Step II

P_5	0	1/4	-60	0	9	1	0	1/25	1/25
P_6	0	1/2	-90	0	3	0	1	1/50	1/50
P_7	$-1/50$	0	0	1	0	0	0	1	1
		3/4	-150	0	-6	0	0	$-1/50$	$-1/50$

$\theta_1 = 1/25,\ Z_1 - c_1 = \frac{3}{4},\ \theta_1(Z_1 - c_1) = 3/100.$

The entering vector is P_1.

Step III

P_5	0	0	-15	0	15/2	1	$-\frac{1}{2}$	3/100	3/100
P_1	$-\frac{3}{4}$	1	-180	0	6	0	2	1/25	1/25
P_3	$-1/50$	0	0	1	0	0	0	1	1
		0	-15	0	$-21/2$	0	$-\frac{3}{2}$	$-1/20$	$-1/20$

The problem has thus been resolved through just three iterations. This method is found to be convenient when atleast two x_i are zero in the solution column of the table.

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