Indian J. pure appl. Math., **11**(8) : 974–982, August 1980

A NOTE ON ADAMS OPERATIONS

MUNIR S. MORSY

Department of Pure Mathematics, Ain Shams University, Cairo, Egypt

(*Received* 21 *September* 1979)

In this paper we construct a strong relationship between the Adams operations (ψ-operations) and γ-operations in the ring of the unitary group $U(n)$.

1. PRELIMINARIES

We collect some known results from Adam (1961) and Morsy (1979) which are used throughout this paper.

1.1 : *The Construction of the Chern Character*

Let W denote the category whose objects are finite CW-complexes and whose morphisms are homotopy classes of continuous maps. For $X \in W$ set:

$B(X)$ = commutative semi-group of isomorphism classes of complex vector bundles over X with addition given by Whitney sum.

$F(X)$ = free Abelian group generated by the isomorphism classes of complex vector bundles over X.

$R(X)$ = subgroup of $F(X)$ generated by all elements of the form:

$$\xi_1 \oplus \xi_2 - \xi_1 - \xi_2,$$

where $\oplus$ indicates the Whitney sum and $\xi_1, \xi_2 \in B(X)$.

Definition 1.1.1 — The Grothendieck group $K(X)$ is defined as the quotient group $F(X)/R(X)$. The operation in $K(X)$ is denoted by $+$.

The operation of forming the tensor product of two bundles makes $K(X)$ into a functor from W to the category of commutative rings with unit.

Definition 1.1.2 — Let $\xi \in B(X)$. The rank of ξ is the homomorphism

$$B(X) \to H^0(X; Z) : \xi \to \dim \text{(fibre of } \xi).$$

Given an element $\xi \in B(X)$. Let

$$c(\xi) = 1 + c_1(\xi) + \ldots \in H^{2i}(X; Z)$$

be the total Chern class of ξ. Then there is an induced homomorphism of Abelian groups $c : K(X) \to G^{even}(X; Z)$, where $G^{even}(X; Z)$ is the set of all sum $1 + h_2 + h_4 + \ldots$, with $h_{2i} \in H^{2i}(X; Z)$ and with group operation given by the cup product in $H^*(X; Z)$.

For $\xi \in B(X)$ define

$$\text{ch}(\xi) = \text{rank}(\xi) + \sum_{i=1} \frac{s_i(c_1(\xi), c_2(\xi), \ldots, c_i(\xi))}{\underline{i|}}$$

where s_i is the universal polynomial.

Definition 1.1.3 — The natural ring homomorphism

$$\text{ch} : K(X) \to H^{even}(X;Q),$$

where Q denotes the field of rationals, is called the Chern character.

1.2 : *Adams Operations and* γ*-operations*

Any compact Lie group G has a countable number of irreducible complex representations, let these be $(\rho_1, \rho_2, \ldots)$.

Definition 1.2.1 — The complex representation ring of G, denoted by R_G, is the free Abelian group generated by ρ_i. It should be noted that,

(i) The tensor product of representations gives R_G the structure of a ring.

(ii) The complex representations correspond to elements of the free Abelian semi-group $\sum_{i \geqq 0} n_i \rho_i$, where $n_i \geqq 0$.

Definition 1.2.2 — A λ-semi-ring is a pair (R_G, λ^i), where R_G is a complex representation ring of G and λ^i are functions of $R_G \to R_G$, for $i \geqq 0$, satisfying the following properties:

Axiom (1) : $\lambda^0(g) = 1$ and $\lambda^1(g) = g$ for each $g \in R_G$.

Axiom (2) : For each

$$g_1, g_2 \in R_G, \lambda^k(g_1 \oplus g_2) = \sum_{i+j=k} \lambda^i(g)_1 \oplus \lambda^j(g_2).$$

Definition 1.2.3 — A λ-ring is a λ-semi-ring whose underlying semi-ring is a ring.

Let $R_G\,[[t]]$ be the formal power series in "t" with coefficients in R_G and let $1 + \tilde{R}_G\,[[t]]$ be the multiplicative group with constant term 1

Definition 1.2.4 — Define $\lambda_t \in 1 + \tilde{R}_G\,[[t]]$ by the relation

$$\lambda_t(g) = \sum_{i \geqq 0} \lambda^i(g)\, t^i \text{ for each } g \in R_G.$$

By axiom (2) we have

$$\lambda_t(g_1 \oplus g_2) = \lambda_t(g_1) \oplus \lambda_t(g_2) \text{ for each } g_1, g_2 \in R_G.$$

Definition 1.2.5 — Let $\psi_t(g) = \sum_{i \geqq 1} \psi^k(g)\, t^k$ be given by the relation

$$\psi_{-t}(g) = -\, t \left(\left(\frac{d}{dt} \right) \lambda_t(g) \right) \Big/ \lambda_t$$

for a λ-ring R_G. The functions

$$\psi^k : R_G \to R_G$$

are called the Adams ψ-operations in R_G.

Lemma 1 — There is the following relation between λ^i and ψ^k for $g \in R_G$:

$$\psi^k(g) - \lambda^1(g)\, \psi^{k-1}(g) + \ldots + (-1)^{k-1} \lambda^{k-1}(g)\, \psi^1(g) + (-1)^k k\lambda^k(g) = 0.$$

PROOF : It follows direct from definition (1.2.5).

Definition 1.2.6 — The γ-operations in R_G, denoted $\gamma^i : R_G \to R_G$, are defined by the requirement tnat

$$\gamma_t(g) = \lambda_{t/(1-t)}(g),$$

where $$\gamma_t(g) = \sum_{i \geqq 0} \gamma^i(g)\, t^i \text{ for } g \in R_G, \text{ i.e.}$$

$$\gamma_t(g) = \sum_{i \geqq 0} \lambda^i(g)\, t^i(1 + it + \ldots).$$

2. The Construction of the Relationship between ψ- and γ-operations

Take $G = U(n)$ and T be the standard maximal torus in $U(n)$. Then

$$R_T = Z\,[x_i, x_i^{-1}],\ i = 1, 2, \ldots, n;$$

where x_i is the ith diagonal element representing the representation

$$t.z = x_i(g) \text{ for } z \in C \text{ and } t \in T.$$

Combining the two definitions (1.2.5) and (1.2.6) and considering the relations between ψ^k and γ^k with λ^k, we have

$$\psi^k(\gamma_t) = \prod_{i=1}^{n} (1 + t\psi^k(x_i)) \qquad \ldots(1)$$

Theorem 1

$$\psi^2 \begin{pmatrix} \gamma^n \\ \gamma^{n-1} \\ \vdots \\ \gamma^1 \end{pmatrix} = \begin{pmatrix} 2^n & 0 & \cdots & \cdots & & \cdots & \cdots & 0 \\ -n2^{n-2} & 2^{n-1} & 0 & \cdots & & \cdots & \cdots & 0 \\ \frac{n(n-3)}{2!} 2^{n-4} & -n(n-1)\, 2^{n-3} & 2^{n-2} & 0 & \cdots & & & 0 \\ & & & & 2^{n-3} & & \cdots & 0 \\ & & & & & \ddots & 2^2 & 0 \\ & & & & & & -2 & 2 \end{pmatrix} \times \begin{pmatrix} \gamma^n \\ \gamma^{n-1} \\ \vdots \\ \gamma^1 \end{pmatrix}$$

$$\left[\frac{n-1}{2}\right] \Big\{$$

The ith eigenvalues are 2^p, $1 \leqslant p \leqslant n$.

PROOF : Let ρ_n be the standard representation in $R_{U(n)}$. Therefore

$$\rho_n \to x_1 + \ldots + x_n.$$

If we write $\lambda^i = \lambda^i(\rho_n - n)$ and since the restricted homomorphism

$$R_{U(n)} \to R_T$$

is an isomorphism, it is preserved after completion with respect to the augmentation topology, onto the ring of invariant under the action of the Weyl group in R_T which acts by permutation of the x_i, then

$$R^c_{U(n)} = Z[\lambda^1, \ldots, \lambda^n] = Z[[\lambda]].$$

Define $\gamma^i(x) = \lambda^i(x + i - 1)$ for $x \in R^c_{U(n)}$. Then write

$$\gamma^i = \gamma^i(\rho_n - n) = \lambda^i(\rho_n - n + i - 1),\ i = 1, 2, \ldots, n.$$

Therefore $R^c_{U(n)} = Z[\gamma^1, \ldots, \gamma^n]$. We are now in a situation to see that

$$\gamma_t = \sum_{i=1}^{n} \gamma^i t^i \in Z[X_1, \ldots, X_n, t],$$

where $X_i = x_i - 1$.

If follows from previous construction that relation (1) can be expressed as

$$\psi^k(\gamma_t) = \prod_{i=1}^{n} \prod_{j=1}^{k} (1 + \alpha_j x_i)$$

$$= \prod_{j=1}^{k} \prod_{i=1}^{n} (1 + \alpha_j x_i)$$

$$= \prod_{j=1}^{k} \gamma_{\alpha_j} = 1 + \sum_{r=1}^{\infty} \alpha_1^r + \alpha_2^r + \dots + \alpha_k^r)\, \gamma^r + \text{decomposable elements} \quad \dots(2)$$

where α_j belongs to the algebraic closure of the ring $R_{U(n)}$ in which the integral domain $Z\,[[t]]$ is embedded.

From the relations (1) and (2), it follows that

$$1 + t\psi^k(x_i) = \prod_{j=1}^{k} (1 + \alpha_j x_i) = 1 + t\,\frac{k}{\underline{1|}}\, x_i + t\,\frac{k(k-1)}{\underline{2|}}\, x_i^2 + \dots + t x_i^k \quad \dots(3)$$

Comparing the coefficients of x_i, we see that

$$\sum_i \alpha_i = tk, \quad \sum_{i+j} \alpha_i \alpha_j = t\,\frac{k(k-1)}{\underline{2|}} \text{ etc.}$$

Therefore the symmetric functions of the α_i belong to $Z\,[[t]]$. Now suppose that α_1 and α_2 are the roots of the quadratic equation $\theta^2 - 2\theta t + t = 0$. Thus

$$(1 - 2ty + ty^2) = (1 - \alpha_1 y)(1 - \alpha_2 y)$$

for all t and for all y. Therefore

$$\frac{1}{r}(\alpha_1^r + \alpha_2^r) = \text{coefficient of } y^r \text{ in the expansion of}$$

$$-\ln\,[1 - ty(2 - y)]$$

for we can find a range of "t" such that α_1 and α_2 are real and choose sufficiently small "y" to make expansion valid. By the usual expansion of $-\ln\,[1 - ty(2 - y)]$, we find that

$$\frac{1}{r}(\alpha_1^r + \alpha_2^r) = \text{coefficient of } y^r \text{ in } 2ty + \frac{(2ty)^2}{2}\left(1 - \frac{y}{2}\right)^2 + \frac{(2ty)^3}{3}(1 - y/2)^3 + \ldots$$

$$= \frac{(2t)^r}{r} - \frac{(2t)^{r-1}}{2(r-1)}\binom{r-1}{1} + \frac{(2t)^{r-2}}{2^2(r-2)}\binom{r-2}{2}$$

$$- \begin{cases} + (-1)^{r/2}(2t)^{r/2}\dfrac{2}{r}\left(\dfrac{1}{2}\right)^{r/2} & \text{if } r \text{ is even} \\ + (-1)^{(r-1)/2}(2t)^{(r+1)/2}\dfrac{2}{r+1}\left(\dfrac{1}{2}\right)^{(r-1)/2}\dbinom{\frac{r+1}{2}}{\frac{r-1}{2}} & \text{if } r \text{ is odd.} \end{cases}$$

Equating the coefficients of t^θ:

$$\psi^2(\gamma^\theta) = \sum_{\theta \leqq p \leqq 2\theta} (-1)^{p-\theta}\frac{p}{\theta}\binom{\theta}{p-\theta}2^{2\theta-p}\gamma^p + \text{decomposable elements.}$$

By neglecting the decomposable elements, we see that the first few terms are

$$\psi^2(\gamma^n) = 2^n\gamma^n,\ \psi^2(\gamma^{n-1}) = 2^{n-1}\gamma^{n-1} - n2^{n-2}\gamma^n,$$

$$\psi^2(\gamma^{n-2}) = 2^{n-2}\gamma^{n-2} - (n-1)\,2^{n-3}\gamma^{n-1} + \frac{n-3}{\underline{2|}}\,2^{n-4}n\gamma^n, \ldots,$$

$$\psi^2(\gamma^2) = 2^2\gamma^2 - 6\gamma^3 + 2\gamma^4,\ \psi^2(\gamma^1) = 2\gamma^1 - 2\gamma^2.$$

Hence the proof is complete.

Now if we write ψ^2 in the form $A\begin{pmatrix} 2^n & & & \\ & 2^{n-1} & & \\ & & \ddots & \\ & & & 2^2 & \\ & & & & 2 \end{pmatrix}A^{-1}$, i.e., A has columns corresponding to eigenvalues $2^n, 2^{n-1}, \ldots, 2$. Then we prove the following theorem:

Theorem 2 — The sth column of A is $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_1(s) \\ \vdots \\ a_s(s) \end{pmatrix}$, for $1 \leqslant s \leqslant n$,

and corresponding to eigenvalue 2^s,

where $a_r(s)$ is the coefficient of t^r in the expansion of $\left\{\frac{\ln(1-t)}{-t}\right\}^{-s}$.

PROOF : If the eigenvector is $\begin{pmatrix} v_n \\ \vdots \\ v_1 \end{pmatrix}$ for eigenvalue 2^s, then

$$2^s v_n = 2^n v_n,\ 2^s v_{n-1} = 2^{n-1} v_{n-1} - n2^{n-2} v_n,\ 2^s v_{n-2} = 2^{n-2} v_{n-2} - (n-1)\, 2^{n-3} v_{n-1} + n \frac{(n-3)}{\underline{2|}} 2^{n-4} v_n \dots \text{etc.}$$

Therefore $v_{s+1} = v_{s+2} = \dots = v_n = 0$. Write $v_s = a_1(s)$, $v_{s-1} = a_2(s)$ ... etc.

Hence wet get

$$2^s a_2(s) = 2^{s-1} a_2(s) - s2^{s-2} a_1(s),\ 2^s a_3(s) = 2^{s-2} a_3(s) - (s-1)\, 2^{s-3} a_2(s) + 2^{s-4} \frac{s-3}{\underline{2|}} s a_1(s), \dots \text{etc.}$$

We continue by induction:

$$a_1(s)\left[\frac{\ln(1-t)}{-t}\right]^{-s} = a_1(s)\left[\sum_{i=0}^{\infty} \frac{t^i}{1+i}\right]^{-s}$$

$$= a_1(s)\left[1 - \frac{s}{2} t + \frac{3s^2 - 5s}{24} t^2 + \dots\right].$$

Now $a_2(s) = -\frac{s}{2} a_1(s)$, and $a_3(s) = \frac{3s^2 - 5s}{24} a_1(s)$. Assume result for $a_i(s)$, where $i < p$. Since

$$2^s a_p(s) = 2^{s-p} a_p(s) - 2^{s-p-1}(s-p+1)\, a_{p-1}(s) + 2^{s-p-2}(s-p+2) \times \frac{s-p-1}{\underline{2|}} a_{p-2}(s),$$

it follows that

$$(s-p)\,(2^p - 1)\, a_p(s) = -\tfrac{1}{2}(s-p+1)\binom{s-p}{1} a_{p-1}(s) + \frac{1}{2^2}(s-p+2)\binom{s-p}{2} a_{p-2}(s).$$

Hence by inductive hypothesis we see that

$$(s-p)\,(2^p - 1)\, a_p(s) = \text{coefficient of } t^p \text{ in } a_1(s)\left[\frac{\ln(1-t)}{-t}\right]^{-s} \times \left[-(s-p+1)\binom{s-p}{1}\left(\frac{t}{2}\right) + (s-p+2)\binom{s-p}{2}\left(\frac{t}{2}\right)^2 + \dots\right]$$

$$= \text{coefficient of } t^p \text{ in } a_1(s) \left[\left(\frac{1}{t^{s-p-1}} \right) \times \frac{d}{dt} \left(t^{s-p} \left(1 - \frac{t}{2} \right)^{s-p} \right) - (s-p) \right] \times \left(\frac{\ln(1-t)}{-t} \right)^{-s} a_1(s).$$

Therefore the result will follow if we can show that

$$\text{coefficient of } t^{n-q} \text{ in } \left(1 - \frac{t}{2}\right)^{q-1} (1-t) \left(\frac{1}{\ln(1-t)} \right)^n = \text{coefficient of } t^{n-q} \text{ in } 2^{n-q} \left(\frac{t}{\ln(1-t)} \right)^n$$

i.e.,

$$\int \frac{1}{z^{n-q+1}} \left[\frac{z^n 2^{n-q} - (1-(z/2))^{q-1}(1-z) z^n}{\ln^n(1-z)} \right] dz = 0$$

where the integral is taken around a small circle and "ln" has principal value $w = \ln(1-z)$, i.e. $z = 1 - e^w$. Now the integral takes the form,

$$\int \frac{e^w(1-e^w) 2^{n-q} - 2^{-q+1}(1+e^w)^{q-1} e^{2w}}{w^n} dw = \int \frac{2^{n-q} e^w (1-e^w)^{q-1}}{w^n} dw - \int \frac{2^{n-q} e^{2w}(1-e^{2w})^{q-1}}{(2w)^n} d(2w) = 0.$$

Take $a_1(s) = 1$, then the theorem is proved.

We are now in position to prove the main theorem

Theorem 3 — $$\psi^p \begin{pmatrix} \gamma^n \\ \vdots \\ \gamma^1 \end{pmatrix} = A \begin{pmatrix} p & & & \\ & p^{n-1} & & \\ & & \ddots & \\ & & & p^2 \\ & & & & p \end{pmatrix} A^{-1} \begin{pmatrix} \gamma^n \\ \vdots \\ \gamma^1 \end{pmatrix}$$

PROOF: The result is clear for $p = 2^n$. Let $A^{-1} = (\bar{a}_{ij})$, $\bar{a}_{ij} = 0$ for $i < j$ since ψ^p is additive, it is enough to show that

$$\psi^p A^{-1} \begin{pmatrix} \gamma^n \\ \vdots \\ \gamma^1 \end{pmatrix} = \begin{pmatrix} p^n & & & \\ & p^{n-1} & & \\ & & \ddots & \\ & & & p^2 \\ & & & & p \end{pmatrix} A^{-1} \begin{pmatrix} \gamma^n \\ \vdots \\ \gamma^1 \end{pmatrix}.$$

Let ch_i be the $2i$-dimensional part of the Chern character "ch". Combining the facts $ch_n \psi^2 = 2^n ch_n$ and $ch_n \psi^p = p^n ch_n$ together with Theorems 1, 2, it follows that

$$\psi^2(\bar{a}_{ij})\begin{pmatrix}\gamma^n\\ \vdots\\ \gamma^1\end{pmatrix} = \begin{pmatrix}2^n & & \\ & \ddots & \\ & & 2\end{pmatrix} A^{-1}\begin{pmatrix}\gamma^n\\ \vdots\\ \gamma^1\end{pmatrix},$$

$$2^s(\bar{a}_{ij})\begin{pmatrix}0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \mathrm{ch}_s\gamma^s & \\ & & & & \ddots \\ & & & & & \mathrm{ch}_s\gamma^1\end{pmatrix} = \begin{pmatrix}2^n & & \\ & \ddots & \\ & & 2\end{pmatrix}(\bar{a}_{ij})\begin{pmatrix}0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \mathrm{ch}_s\gamma^s & \\ & & & & \ddots \\ & & & & & \mathrm{ch}_s\gamma^1\end{pmatrix}.$$

Therefore

$$2^s \sum_{j=1}^{n} \bar{a}_{tj}\, \mathrm{ch}_s\, \gamma^{n-j+1} = 2^t \sum_{j=1}^{n} \bar{a}_{tj}\, \mathrm{ch}_s\, \gamma^{n-j+1}$$

where $t = 1, 2, \ldots, n$.

But since

$$\sum_{j=1}^{n} \bar{a}_{tj}\, \mathrm{ch}_s\, \gamma^{n-j+1} = 0 \quad \text{unless } t = s$$

it follows that

$$p^s \sum_{j=1}^{n} \bar{a}_{tj}\, \mathrm{ch}_s\, \gamma^{n-j+1} = p^t \sum_{j=1}^{n} \bar{a}_{tj}\, \mathrm{ch}_s\, \gamma^{n-j+1} \quad \text{for } t = 1, 2, \ldots, n.$$

Hence

$$\mathrm{ch}_s\, \psi^p(\bar{a}_{ij})\begin{pmatrix}\gamma^n\\ \vdots\\ \gamma^1\end{pmatrix} = \mathrm{ch}_s\begin{pmatrix}p^n & & \\ & \ddots & \\ & & p\end{pmatrix}(\bar{a}_{ij})\begin{pmatrix}\gamma^n\\ \vdots\\ \gamma^1\end{pmatrix} \quad \text{for all } s$$

i.e.,

$$\mathrm{ch}\, \psi^p A^{-1}\begin{pmatrix}\gamma^n\\ \vdots\\ \gamma^1\end{pmatrix} = \mathrm{ch}\begin{pmatrix}p^n & & \\ & \ddots & \\ & & p\end{pmatrix} A^{-1}\begin{pmatrix}\gamma^n\\ \vdots\\ \gamma^1\end{pmatrix}.$$

This completes the proof of the theorem.

References

Adams, J. F. (1961). On Chern characters and the structure of the unitary group. *Proc. Camb. phil. Soc.*, **57**, 189–99.

Morsy, M. S. (1979). Riemann-Roch theorem and the Cobordism theory. *Egyptian Computer Sci. J.*, **2**, No. 1.