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# ON HARMONIC AND KILLING VECTOR FIELDS IN A SUBMANIFOLD

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In this paper, the author obtains a characterization for the first Betti number of submanifold of a space of constant curvature to vanish in terms of the principal curvatures.

## INTRODUCTION

Bochner (1949), Couty (1958), Myers (1941), Yano (1952) and others have investigated conditions for a Riemannian manifold to admit harmonic vector fields, projective Killing vector fields and conformal Killing vector fields. The purpose of this paper is to continue the above study and obtain a characterization for the first Betti number of submanifold of a space of constant curvature to vanish in terms of the principal curvatures.

## 1. PRELIMINARIES

Let $M$ be a closed orientable differentiable manifold of dimension $n$ imbedded in an orientable $m$-dimensional Riemannian manifold $N$ of constant sectional curvature $k$. Let $g$ be the Riemannian metric induced on $M$. Let $\bar{\nabla}$ and $\nabla$ be the Riemannian connections on $N$ and $M$ respectively, then

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y)$$

for vector fields $X$, $Y$ on $M$, where $H$ is the second fundamental form of $M$ in $N$. Let $e$ be a unit normal field on $M$. We may decompose $\bar{\nabla}_X e$ as

$$\bar{\nabla}_X e = - A_e(X) + \nabla^{\perp}_X e$$

where $-A_e(X)$ and $\nabla^{\perp}_X e$ are respectively the tangential and normal components of $\bar{\nabla}_X e$. $A_e$ is called the Weingarten map of $M$ relative to the normal field $e$. $A_e(X)$ is bilinear in $e$ and $X$. Let $e_1, e_2, \ldots, e_{m-n}$ form an orthonormal basis in the normal bundle of $M$ in $N$ and $h^x$ be the second fundamental form corresponding to $e_x$, where, and in the sequel, $x$ runs over the range $1, 2, \ldots, m-n$. By the Gauss equation we have (see Chen (1973))

$$R(X, Y) = k(n-1)\, g(X, Y) + p_1^x\, h_x(X, Y) - g(A^x(X), A_x(Y))$$

consequently

$$R(X, X) = k(n-1)\,\| X \|^2 + p_1^x\, h_x(X, X) - \Sigma_x \| A_x(X) \|^2 \qquad \ldots(1.1)$$

where $R$ is the Ricci tensor of $M$, $A_x = A^x$ is the Weingarten map relative to $e_x$, $\| \ \|$ is the norm induced by the metric $g$, $p_1^x$ = trace $A^x$ and the Einstein's summation convention is assumed.

$$\bar{e} = \frac{1}{n} p_1^x\, e_x$$

is called the mean curvature vector. $M$ is said to be minimal if $\bar{e}$ vanishes identically. In that case $p_1^x$ vanishes. Let $v_1^x, \ldots, v_n^x$ be orthonormal frame of principal directions of the normal direction $e_x$ and $k_{x1}, \ldots, k_{xn}$ be the corresponding principal curvatures. Then we have from (1.1)

$$R(X, X) = k(n-1)\,\| X \|^2 + \sum_{i,j\, i \neq j} k_{xi} k_{xj}\, | g(X, v_i^x) |^2 \qquad \ldots(1.2)$$

where $i, j$ run over the range $1, \ldots, n$.

For a vector field $X$ on $M$ we have (See Yano 1970))

$$\int_M \{R(X, X) - \tfrac{1}{2} \| d\xi \|^2 + \| \nabla X \|^2 - (\delta\xi)^2\}\, dv = 0 \qquad \ldots(1.3)$$

$$\int_M \{R(X, X) + \tfrac{1}{2} \| L_X g \|^2 - \| \nabla X \|^2 - (\delta X)^2\}\, dv = 0 \qquad \ldots(1.4)$$

$$\int_M \{g(\triangle X, X) - \tfrac{1}{2} \| d\xi \|^2 - (\delta X)^2\}\, dv = 0 \qquad \ldots(1.5)$$

where $\xi$ is the one form associated with $X$, $d$ is the exterior derivative operator, $\delta$ is the codifferential operator, $L_X g$ is the Lie derivative of the metric $g$ with respect to the vector field $X$ and $\triangle X$ is the vector field associated with the one form

$$\triangle\, \xi = (\delta d + d\delta)\, \xi.$$

From (1.2), (1.3) and (1.4) we have

$$\int_M \{\tfrac{1}{2} \| d\xi \|^2 + (\delta\xi)^2\}\, dv$$

$$= \int_M \{k(n-1)\,\| X \|^2 + \| \nabla X \|^2 + \sum_{i,\, j\, i \neq j} k_{xi} k_{xj}\, | g(X, v_i^x) |^2\}\, dv$$

$$\tfrac{1}{2} \int_M \| L_X g \|^2\, dv \qquad \ldots(1.6)$$

$$= \int_M \{\| \nabla X \|^2 + (\delta X)^2 - k(n-1)\,\| X \|^2 - p_1^x\, h_x(X, X) + \Sigma_x \| A_x(X) \|^2\}\, dv. \qquad \ldots(1.7)$$

We need the following theorem.

*Theorem* (Hodge 1951) — In a compact orientable Riemannian manifold the number of linearly independent (with constant real coefficients) harmonic $p$-forms is equal to the $p$th-dimensional Betti number of the manifold.

## 2. Harmonic Vector Fields

A vector field $X$ in $M$ or the one form $\xi$ associated with it is said to be harmonic if

$$d\xi = 0 \quad \text{and} \quad \delta\xi = 0. \qquad \ldots(2.1)$$

From (1.6) and (2.1) we have

*Proposition* 2.1 — For a vector field $X$ in $M$

$$\int_M \{k(n-1) \| X \|^2 + \| \nabla X \|^2 + \sum_{i, j, i \neq j} k_{xi} k_{xj} \mid g(X, v_i^x) \mid^2\} dv \geqslant 0$$

equality holding if and only if $X$ is a harmonic vector field.

As an immediate consequence we have the following:

*Proposition* 2.2 — In a closed orientable submanifold $M$ of a space of non-negative constant curvature $k$ if each of the principal curvatures $k_{xi}$ relative to normals $e_x$ in an orthonormal frame $(e_x)$ in the normal bundle are non-negative (non-positive) then every harmonic vector field $X$ in $M$ is parallel. Further if each $k_{xi}$ is positive (negative) or if $k$ is positive then harmonic vector field other than zero does not exist on $M$, consequently, the first Betti number vanishes.

*Corollary* 2.3 — If the Gaussian curvature $K$ of a compact orientable surface $M$ in $E^3$ is non-negative, then every harmonic vector field $X$ in $M$ is parallel in $M$. Further if $K$ is positive then harmonic vector field other than zero does not exist on $M$, consequently, the first Betti number vanishes in $M$.

## 3. Projective Killing Vector Field

A vector field $X$ in $M$ is said to be projective Killing if

$$(L_X \nabla)(Y, Z) = \pi(Y) Z + \pi(Z) Y \qquad \ldots(3.1)$$

for vector fields $Y, Z$ in $M$, $\pi$ being certain one form given by

$$d(\delta X) = -(n+1)\, \pi. \qquad \ldots(3.2)$$

(3.1) and (3.2) together imply

$$\triangle X - 2QX - \frac{2}{n+1} d(\delta X) = 0 \qquad \ldots(3.3)$$

where $Q$ is the tensor field of type (1, 1) associated with the Ricci tensor. Substituting (3.3) in (1.5) we have (see Yano 1970)

$$\int_M \left\{2R(X, X) - \tfrac{1}{2} \| d\xi \|^2 - \frac{n-1}{n+1} (\delta X)^2\right\} dv = 0. \qquad \ldots(3.4)$$

Now making use of (1.1) we have

*Proposition* 3.1 — Let $X$ be a projective Killing vector field on a closed orientable submanifold $M$ of a space of constant curvature $k$. Then

$$\int_M \left\{\tfrac{1}{4} \| d\xi \|^2 + \frac{n-1}{2(n+1)} (\delta X)^2 - p_1^x h_x(X, X) - k(n-1) \| X \|^2 + \sum_x \| A_x(X) \|^2\right\} dv = 0.$$

Since

$$\Sigma_x \| A_x(X) \|^2 - k(n-1) \| X \|^2 = \Sigma_{x,i} \{(k_{xi})^2 - k(n-1)/m\} \mid g(X, v_i^x) \mid^2 \qquad \ldots(3.5)$$

we have from Proposition 3.1 the following:

*Proposition* 3.2 — Let $M$ be a closed orientable minimal submanifold of a space of constant curvature $k$. If $(k_{xi})^2 > k(n-1)/m$ for each $x$ and for each $i$, then the projective Killing vector field other than zero does not exist on $M$.

## 4. Conformal Killing Vector Field

A vector field $X$ on $M$ is said to be conformal Killing if

$$L_X g = 2\rho g \qquad \ldots(4.1)$$

where $\rho$ is a function given by

$$\rho = -\frac{1}{n} \delta X. \qquad \ldots(4.2)$$

Making use of (4.1) and (4.2) in (1.7) we have

*Proposition* 4.1 — Let $X$ be a conformal Killing vector field on a closed orientable submanifold $M$ of a space of constant curvature $k$. Then

$$\int_M \| \nabla X \|^2 + \frac{n-2}{n} (\delta X)^2 - k(n-1) \| X \|^2 - p_1^x h_x(X, X) \times \sum_x \| A_x(X) \|^2\Big\} dv = 0.$$

Hence from (3.5) we have the following:

*Proposition* 4.2 — Let $M$ be a closed orientable minimal submanifold of a space of constant curvature $k$. If $(k_{xi})^2 \geqslant k(n-1)/m$ for each $x$ and for each $i$ then every conformal Killing vector field is parallel in $M$. Further if the above inequality is strict then conformal Killing vector field other than zero does not exist on $M$.

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