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SUBSPACES OF A FINSLER SPACE ADMITTING CONCURRENT VECTOR FIELD

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In the present paper, the author studies subspaces of a Finsler space admitting concurrent vector field.

1. INTRODUCTION

Sasaki (1942) and Yano (1943) have studied concurrent vector field on a Riemannian manifold. Matsumoto and Eguchi (1974) have discussed the concurrent vector field on a Finsler space. In the present paper, we study the subspaces of a Finsler space admitting a concurrent vector field.

2. NOTATIONS AND BASIC CONCEPTS

Consider a Finsler space F_n of n dimensions, referred to a local coordinate system $x^i (i = 1, 2, 3, \ldots, n)$, whose metric function $F(x^i, \dot{x}^i)$ satisfies the conditions usually imposed upon it (Rund 1959, Chap. I). The metric tensor of F_n is defined by

$$g_{ij}(x, \dot{x}) = \tfrac{1}{2}\, \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $\dot{\partial}_i = \dfrac{\partial}{\partial \dot{x}^i}$.

A vector field X^i is said to be concurrent on the Finsler space F_n if the following conditions are satisfied:

$$X^i \text{ is independent of } \dot{x} \qquad \ldots(2.1)$$

$$X^i C_{ijk} = 0 \qquad \ldots(2.2)$$

$$X^i_{\ |j} = -\delta^i_j \text{ (or say } X_{i|j} = -g_{ij}) \qquad \ldots(2.3)$$

where $X^i_{\ |j}$ is the Cartan's covariant derivative of X^i and $C_{ijk}(x, \dot{x}) = \tfrac{1}{2}\, \dot{\partial}_k g_{ij}(x, \dot{x})$.

A subspace F_m, of F_n, may be represented parametrically in the form $x^i = x^i(u^\alpha)$, $(\alpha = 1, 2, 3, \ldots, m)$, where u^α are the Gaussian coordinates of F_m. It will be assumed throughout the paper that the functions x^i are at least of class C^3 and the matrix of the projection parameters $B^i_\alpha = \partial_\alpha x^i$ has rank m. The following notations will be used here.

$$B^i_{\alpha\beta} = \partial_\alpha \partial_\beta x^i, \; B^{ij\cdots k}_{\alpha\beta\cdots\gamma} = B^i_\alpha B^j_\beta \ldots B^k_\gamma .$$

Since the rank of the matrix $\| B^i_\alpha \|$ is assumed to be m, it follows that there exists a field of $(n - m)$ linearly independent vectors $N^i_{(\nu)}$ normal to F_m and they may be represented by the relations,

$$N_{(\mu)i} B^i_\alpha = g_{ij} N^j_{(\mu)} B^i_\alpha = 0, \; (\mu = m + 1, \ldots, n). \qquad \ldots(2.4)$$

These vectors are normalized by means of the relations,

$$g_{ij}(x, \dot{x}) \, N^i_{(\mu)} N^j_{(\nu)} = \delta_{\mu\nu}$$

$$N^i_{(\mu)} = g^{ij}(x, \dot{x}) \, N_{(\mu)j}.$$

The induced connection parameter $\Gamma^{*\alpha}_{\beta\gamma}$, of F_m, is given by,

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i \, (B^i_{\beta\gamma} + \Gamma^{*i}_{hk} \, B^{hk}_{\beta\gamma}). \qquad \ldots(2.5)$$

This connection parameter is used in defining the following induced mixed derivative of a tensor of the type T^i_α.

$$T^i_{\alpha\|\beta} = \partial_\beta T^i_\alpha - \dot{\partial}_\gamma T^i_\alpha \, \Gamma^{*\gamma}_{\beta\rho} \, \dot{u}^\rho + T^\gamma_\alpha \, \Gamma^{*i}_{j\gamma} \, B^j_\beta - T^i_\rho \, \Gamma^{*\rho}_{\alpha\beta} \qquad \ldots(2.6)$$

In particular, we have

$$I^i_{\alpha\beta} \overset{def}{=} B^i_{\alpha\|\beta} = B^i_{\alpha\beta} + \Gamma^{*i}_{hk} \, B^h_\alpha \, B^k_\beta - \Gamma^{*\epsilon}_{\alpha\beta} \, B^i_\epsilon \qquad \ldots(2.7)$$

which regarded as vector of the imbedding space F_n, is normal to F_m. Hence we may write

$$I^i_{\alpha\beta} = \sum_\mu N^i_{(\mu)} \, \bar{\Omega}_{(\mu)\alpha\beta} \qquad \ldots(2.8)$$

where $\bar{\Omega}_{(\mu)\alpha\beta}(u, \dot{u})$ are the components of the second fundamental tensors of F_m. The induced derivative $N^i_{(\mu)\|\beta}$ of the type (2.6) is given by Srivastava and Sinha (1969) in the form,

$$N^i_{(\mu)\|\gamma} = - \bar{\Omega}_{(\mu)\alpha\gamma} g^{\alpha\delta} B^i_\delta + E^i_{(\mu)l} \, I^l_{\rho\gamma} \, \dot{u}^\rho \qquad \ldots(2.9)$$

where

$$E^i_{(\mu)l} = \sum_\sigma N^i_{(\sigma)} M_{(\mu\sigma)l} - 2M^i_{(\mu)l}. \quad ...(2.10)$$

and

$$\left.\begin{aligned} M^i_{(\mu)l} &= C^i_{lp} N^p_{(\mu)}, \\ M_{(\mu\sigma)l} &= C_{pkl} N^p_{(\mu)} N^k_{(\sigma)} = M_{(\mu)kl} N^k_{(\sigma)}. \end{aligned}\right\} \quad ...(2.11)$$

3. Concurrent Vector Field on the Subspace

Consider a congruence of curves such that one curve of the congruence passes through every point of F_m. The contravariant components X^i, of a vector field in the direction of curves of the congruence can be expressed in the form,

$$X^i = X^\alpha B^i_\alpha + \sum_\nu d_{(\nu)} N^i_{(\nu)}. \quad ...(3.1)$$

Carrying out the mixed induced covariant derivative of (3.1) and simplifying with the help of eqns. (2.7) and (2.9) we get

$$X^i_{\|\gamma} = q^\alpha_\gamma B^i_\alpha + \sum_\nu \nu_{(\nu)\gamma} N^i_{(\nu)} + \sum_\nu E^i_{(\nu)l} I^l_{\rho\gamma} \dot{u}^\rho d_{(\nu)} \quad ...(3.2)$$

where

$$q^\alpha_\gamma \equiv X^\alpha_{|\gamma} - \sum_\nu \bar{\Omega}_{(\nu)\delta\gamma} g^{\delta\alpha} d_{(\nu)}$$

$$\nu_{(\nu)\gamma} \equiv X^\alpha \bar{\Omega}_{(\nu)\alpha\gamma} + d_{(\nu)\|\gamma}.$$

If the vector field X^i is tangential to the hypersurface then $d_{(\nu)} = 0$ and $X^i = B^i_\alpha X^\alpha$. Equation (3.2) gives

$$X^i_{\|\gamma} = X^\alpha_{|\gamma} B^i_\alpha + \sum_\nu \bar{\Omega}_{(\nu)\alpha\gamma} X^\alpha N^i_{(\nu)}. \quad ...(3.3)$$

But

$$X^i_{\|\gamma} = X^i_{|j} B^j_\gamma + \frac{\partial X^i}{\partial \dot{x}^l} I^l_{\sigma\gamma} \dot{u}^\sigma. \quad ...(3.4)$$

Using conditions (2.1) and (2.3) for the concurrent vector field and eqns. (3.3) and (3.4) we get

$$(X^\alpha_{|\gamma} + \delta^\alpha_\gamma) B^i_\alpha + \sum_\nu \bar{\Omega}_{(\nu)\alpha\gamma} X^\alpha N^i = 0. \quad ...(3.5)$$

Hence we have the following theorem:

Theorem 1 — If a vector field tangent to the subspace is concurrent on the embedding space then it is concurrent with respect to the induced connection parameter in the subspace and the relation $\bar{\Omega}_{(\nu)\alpha\gamma}X^{\alpha} = 0$ (for $\nu = 1, 2, ..., n$), holds.

A relation between the intrinsic and induced connection parameters $'\Gamma^{*}_{\alpha\beta\gamma}$ and $\Gamma^{*}_{\alpha\beta\gamma}$ of the subspace F_m, of the Finsler space F_n, is given in the following form (Rund 1959):

$$'\Gamma^{*}_{\alpha\beta\gamma} - \Gamma_{\alpha\beta\gamma} = \sum_{\mu=m+1} N^{j}C_{hkj}\{(B^{hk}_{\beta\gamma}\,\bar{\Omega}_{(\mu)\alpha\epsilon} + B^{hk}_{\alpha\beta}\,\bar{\Omega}_{(\mu)\gamma\epsilon}$$
$$- B^{hk}_{\gamma\alpha}\,\bar{\Omega}_{(\mu)\beta\epsilon})\,\dot{u}^{\epsilon} - (C^{\delta}_{\beta\gamma}\,B^{hk}_{\delta\alpha} + C^{\delta}_{\alpha\beta}\,B^{hk}_{\delta\gamma}$$
$$- C^{\delta}_{\gamma\alpha}\,B^{hk}_{\delta\beta})\,\bar{\Omega}_{(\mu)\lambda\epsilon}\,\dot{u}^{\epsilon}\dot{u}^{\lambda}\}. \qquad ...(3.6)$$

Since the vector field is concurrent on F_n, the eqn. (3.6), (2.2) and Theorem 1 give,

$$'\Gamma^{*}_{\alpha\beta\gamma}\,X^{\alpha} = \Gamma^{*}_{\alpha\beta\gamma}\,X^{\alpha}. \qquad ...(3.7)$$

The relation (3.7) and the Theorem 1 give the following:

Theorem 2 — If a vector field tangent to the subspace is concurrent in F_n, then it is concurrent in F_m, with respect to induced and intrinsic connection parameters.

4. Osculating Riemannian Metric Defined by Concurrent Vector Field

Consider a vector field $X^{i}(x)$ of F_n. The Riemannian space V_n defined by the metric tensor $\bar{g}_{ij} = g_{ij}(x, x(u))$ is called osculating Riemannian space. The Cartan's connection parameter of the Finsler space F_n is given by (Sasaki 1942)

$$\Gamma^{i}_{jk}(x, X) = \left\{ {i \atop jk} \right\} + C^{i}_{jk}(x, X) \qquad ...(4.1)$$

where $\left\{ {i \atop jk} \right\}$ are the Christoffel's symbol of V_n. Since the spaces V_n and F_n are referred with respect to the same coordinate system, the equations $x^{i} = x^{i}(u^{\alpha})$, $(i = 1, 2, ..., n;\ \alpha = 1, 2, 3, ..., m)$ represent the subspaces of both the space. Let us denote these subspaces by V_m and F_m and assume that X^{i} be a vector field tangent to F_m. We note that

$$\bar{g}_{\alpha\beta}(u) = g_{\alpha\beta}(u, X(u)). \qquad ...(4.2)$$

This shows that V_m is an osculating Riemannian space of F_m.

The unit vector N^i satisfying the relation $g_{ij}(x, X)\, N^i B^j_\beta = 0$ is normal to the subspaces V_m and F_m. Simplifying eqn. (4.1) with the help of eqn. (2.5) we get

$$\Gamma^{*\alpha}_{\beta\gamma}(u, X) = \left\{ {\alpha \atop \beta\gamma} \right\} + C^{\alpha}_{\beta\gamma}(u, X). \qquad \ldots(4.3)$$

If the vector field X^i, of the subspace, is concurrent on F_n, then by virtue of Theorem 2, it is concurrent on F_m, with respect to induced and intrinsic connection parameters. Therefore the intrinsic connection parameter $'\Gamma^{\alpha}_{\beta\gamma}(u, X)$ will satisfy a relation of the type (4.1). Comparing this relation with (4.3) we get,

$$\Gamma^{*\alpha}_{\beta\gamma}(u, X) = {'\Gamma}^{\alpha}_{\beta\gamma}(u, X).$$

It may be noted that $\Gamma^{*\alpha}_{\beta\gamma}(u, \dot{u})$ is not, in general, equal to $'\Gamma^{\alpha}_{\beta\gamma}(u, \dot{u})$.

A simplification based on eqns. (2.7), (4.1) and (4.3) gives

$$\underset{F}{I^i_{\alpha\beta}}(u, X) = \underset{R}{I^i_{\alpha\beta}}(u) + \Sigma_\mu M_{(\mu)\alpha\beta} \underset{\mu}{N^i} \qquad \ldots(4.4)$$

where

$$M_{(\mu)\alpha\beta} = C_{ijh} B^i_\alpha B^j_\beta N^h_{(\mu)}$$

and the equations $\underset{R}{I^i_{\alpha\beta}} = B^i_{\alpha\|\beta}$, $\underset{R}{I^i_{\alpha\beta}}(u) = B^i_{\alpha;\beta}$ have been obtained in the Finsler space F_n and its osculating Riemannian space V_n respectively.

Equation (4.4) gives

$$\Omega_{(\nu)\alpha\beta}(u, X) = d_{(\nu)\alpha\beta}(u) + M_{(\nu)\alpha\beta}(u, X) \qquad \ldots(4.5)$$

where $\Omega_{(\nu)\alpha\beta}(u, X)$ and $d_{(\nu)\alpha\beta}(u)$ are the components of second fundamental tensor of F_m and V_m respectively. The relation $C_{ijk}X^k = 0$ yields

$$M_{(\nu)\alpha\beta} X^\alpha = 0 \text{ for } \nu = m + 1, \ldots, n.$$

Therefore Theorem 1 and eqn. (4.5) give the following theorems.

Theorem 3 — If a vector field X^α of F_m is concurrent on F_n then any direction of the subspace is conjugate (relative to any normal $\underset{\nu}{N^i}$), in F_m and its osculating Riemannian space V_m, with respect to X^α.

Theorem 4 — If the vector field X^α of F_m is concurrent on F_n then it is along the asymptotic line of F_n and its osculating Riemannian space V_m.

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