Indian J. pure appl. Math., 11(8) : 994–1006, August 1980

AN INEQUALITY FOR INDEFINITE TERNARY QUADRATIC FORMS OF TYPE (2, 1)

R. J. HANS-GILL AND MADHU RAKA

Centre for Advanced Study in Mathematics, Panjab University, Chandigarh 160014

(*Received* 1 *March* 1980)

Let $Q(x, y, z)$ be a real indefinite quadratic form of determinant $D < 0$. Here a positive function $f(t)$, $0 \leqslant t < 1$, is obtained such that given any real numbers x_0, y_0, z_0 there exist integers x, y, z such that

$$t(f(t) \mid D \mid)^{1/3} < Q(x + x_0, y + y_0, z + z_0) \leqslant (f(t) \mid D \mid)^{1/3}.$$

The result is best possible for $t = 0$ and 1/9.

1. INTRODUCTION

Let $Q(x, y)$ be a real indefinite binary quadratic form of discriminant $\Delta^2 > 0$. Blaney (1950) has obtained a function $f(\nu)$, $0 \leqslant \nu < 1$, such that given any real numbers x_0, y_0 there exist integers x, y satisfying

$$\nu^2 f(\nu)\, \Delta < Q(x + x_0, y + y_0) \leqslant f(\nu)\, \Delta. \quad ...(1.1)$$

An analogous result was obtained for real ternary quadratic forms of type (1, 2) by the authors (Hans-Gill and Madhu Raka 1980). Here we take up the case of real ternary quadratic forms of type (2, 1). More precisely we prove the following:

Theorem — Let $Q(x, y, z)$ be a real indefinite ternary quadratic form of type (2, 1) and determinant $D(< 0)$. Let $0 \leqslant t < 1$ be a real number. Then given any real numbers x_0, y_0, z_0 we can find integers x, y, z such that

$$t\,(f(t) \mid D \mid)^{1/3} < Q(x + x_0, y + y_0, z + z_0) \leqslant (f(t) \mid D \mid)^{1/3} \quad ...(1.2)$$

where

$$f(t) = \frac{4}{(1 - t)^2\,(1 - 5t)} \qquad \text{for } 0 \leqslant t \leqslant \tfrac{1}{9} \quad ...(1.3)$$

and

$$f(t) = \frac{1}{(1 - t)^4} \frac{(\sqrt{7 + t} + \sqrt{9t - 1})^3}{(\sqrt{7 + t} + 3\sqrt{9t - 1})} \qquad \text{for } \tfrac{1}{9} \leqslant t < 1. \quad ...(1.4)$$

The sign of equality in (1.2) is necessary if and only if $t = 0$ or $t = \frac{1}{9}$. For $t = 0$, it occurs if and only if $Q \sim \rho\, Q_1$ or $\rho\, Q_2$ where $Q_1 = x^2 + yz$, $Q_2 = x^2 + y^2 - 2z^2$ and $\rho > 0$. While for $t = \frac{1}{9}$, the sign of equality in (1.2) is necessary if and only if $Q \sim \rho\, Q_3$ or $\rho\, Q_4$, $\rho > 0$ where $Q_3 = x^2 + 2yz$ and $Q_4 = x^2 + y^2 - z^2$. Further for

Q_1, Q_2, Q_3, and Q_4 equality is needed in (1.2) if and only if (x_0, y_0, z_0) is congruent to $(0, 0, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ modulo 1, respectively.

2. Some Lemmas

In the course of the proof we shall use the following lemmas:

Lemma 1 — Let $Q(x, y, z)$ be as in the theorem. Then there exist integers u, v, w such that

$$0 < Q(u, v, w) \leqslant (4 \mid D \mid)^{1/3}. \qquad \ldots(2.1)$$

Equality holds in (2.1) if and only if $Q \sim \rho\, Q_1$, $\rho > 0$.

This is a theorem of Davenport (1949).

Lemma 2 — Let α, β, r be real numbers with $r > 1$. Then given any real number x_0, there exists $x \equiv x_0 \pmod 1$ such that

$$0 < (x + \alpha)^2 - \beta^2 \leqslant r \qquad \ldots(2.2)$$

provided

$$\beta^2 \begin{cases} \leqslant \left(\dfrac{r-1}{2}\right)^2 & \text{if } r \text{ is an integer} \\ < \left(\dfrac{[r]}{2}\right)^2 & \text{if } r \text{ is not an integer.} \end{cases} \qquad \ldots(2.3)$$

Further strict inequality in (2.3) implies strict inequality in (2.2). This is Lemma 6 of Dumir (1968).

Let $\phi(y, z)$ be an indefinite binary quadratic form with discriminant $\Delta^2 > 0$.

Lemma 3 — Let $0 < \lambda \leqslant 1$ be a real number. Then given any real numbers y_0, z_0, there exist $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\frac{\Delta}{4\lambda} \leqslant \phi(y, z) < \frac{\Delta\lambda}{4}\,. \qquad \ldots(2.4)$$

This is Lemma 3 of Davenport (1948) and Theorem 1 of Blaney (1950).

Lemma 4 — Let $0 \leqslant \mu \leqslant \frac{1}{3}$. Then for any real numbers y_0, z_0, we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\frac{\mu\,\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}} \leqslant \phi(y, z) < \frac{\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}} \qquad \ldots(2.5)$$

The sign of equality in (2.5) is needed if and only if $\mu = 0$ or

$$\mu = \frac{1}{4m-1}, \quad m = 1, 2, 3, \ldots .$$

For $\mu = 0$, it is needed if and only if $\phi \sim \rho\,\phi_1 = \rho\, yz$ or $\rho\,\phi_2 = \rho(y^2 - z^2)$, $\rho > 0$. For $\mu = \dfrac{1}{4m-1}$, equality in (2.5) is needed if and only if

$$\phi \sim \rho\,\phi'_m = \rho(my^2 - (m+2)z^2).$$

For ϕ_1 equality is required in (2.5) if and only if $(y_0, z_0) \equiv (0, 0) \pmod 1$ and for ϕ_2 and ϕ'_m it is required if and only if $(y_0, z_0) \equiv (\frac{1}{2}, \frac{1}{2}) \pmod 1$.

This is Theorem 2 of Blaney (1950).

Lemma 5 — Let $0 \leqslant \nu < 1$. Given any real numbers y_0, z_0 there exist (y, z) $(y, z) = (y_0, z_0) \pmod 1$ such that

$$\frac{\nu^2 \Delta}{\{(1-\nu)^3 (1+3\nu)\}^{1/2}} \leqslant \phi(y, z) < \frac{\Delta}{\{(1-\nu)^3 (1+3\nu)\}^{1/2}}. \qquad \ldots(2.6)$$

The sign of equality is needed in (2.6) if and only if

either $\nu = 0$ and $\phi \sim \rho\phi_1$ or $\rho\phi_2$ and (y_0, z_0) is as stated in Lemma 4.

or $\nu = \frac{1}{3}$, $\phi(y, z) \sim \rho(3y^2 - z^2)$

or $\nu = \frac{1}{2}$, $\phi(y, z) \sim \rho(y^2 + yz - z^2)$, where $\rho > 0$.

For $3y^2 - z^2$ equality is needed if and only if $(y_0, z_0) \equiv (\frac{1}{2}, \frac{1}{2}) \pmod 1$ and for $y^2 + yz - z^2$ it is needed if and only if $(y_0, z_0) \equiv (0, 0) \pmod 1$.

This is Theorem 3 of Blaney (1950).

3. Proof of the Theorem

Let

$$m = \operatorname*{Inf}_{\substack{u,\, v,\, w \text{ integers} \\ Q(u, v, w) > 0}} Q(u, v, w)$$

Then by Lemma 1, we have

$$0 \leqslant m \leqslant (4\,|\,D\,|\,)^{1/3}. \qquad \ldots(3.1)$$

If $m = 0$, the result follows from a theorem of Watson (1960). So we can now suppose that $m > 0$.

Let $0 < \epsilon_0 < \dfrac{1}{16}$. By Lemma 1, we can find integers u, v, w such that

$$Q(u, v, w) = \frac{m}{1-\epsilon} \leqslant (4\,|\,D\,|\,)^{1/3} \qquad \ldots(3.2)$$

where $0 \leqslant \epsilon < \epsilon_0$ and equality holds only if $Q \sim \rho Q_1$. Further by definition of m, we must have g.c.d. $(u, v, w) = 1$. By a suitable unimodular transformation we can suppose that

$$Q(1, 0, 0) = \frac{m}{1-\epsilon}$$

and write

$$Q(x, y, z) = \frac{m}{1-\epsilon}\{(x + hy + gz)^2 + \phi(y, z)\}$$

where $|h| \leqslant 1/2$, $|g| < 1/2$ and where $\phi(y, z)$ is an indefinite quadratic form of discriminant

$$\Delta^2 = 4 \ |D| \ / \left(\frac{m}{1-\epsilon}\right)^3 \geqslant 1.$$

$\Delta^2 = 1$ if and only if $Q \sim \rho Q_1$. Also by definition of m, for integers x, y, z either $Q(x, y, z) \leqslant 0$ or $Q(x, y, z) \geqslant m$ holds. Because of homogeneity it suffices to prove the following theorem.

Theorem A — Let $Q(x, y, z) = (x + hy + gz)^2 + \phi(y, z)$ where $\phi(y, z)$ is an indefinite binary quadratic form of discriminant

$$\Delta^2 = 4 \ |D| \ \geqslant 1 \ (\Delta^2 = 1 \text{ if and only if } Q \sim Q_1) \quad \text{...(3.3)}$$

and

$$|h| \leqslant 1/2, \ |g| \leqslant 1/2. \quad \text{...(3.4)}$$

Let $0 \leqslant \epsilon < \frac{1}{16}$ and suppose that for integers x, y, z we have either

$$Q(x, y, z) \leqslant 0 \text{ or } Q(x, y, z) \geqslant 1 - \epsilon. \quad \text{...(3.5)}$$

Let

$$d = (f(t) \ |D| \ )^{1/3}. \quad \text{...(3.6)}$$

Then given any real numbers x_0, y_0, z_0, there exist $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ satisfying

$$td < Q(x, y, z) \leqslant d. \quad \text{...(3.7)}$$

The sign of equality in (3.7) is needed if and only if $t = 0$ or $t = \frac{1}{9}$. For $t = 0$ it is needed if and only if $Q \sim Q_1$ or Q_2 while for $t = \frac{1}{9}$ it is needed if and only if $Q \sim Q_3$ or Q_4. For these forms (x_0, y_0, z_0) is as given in the theorem.

Proof of Theorem A

Let

$$h(t) = \frac{(\sqrt{7+t} + \sqrt{9t-1})^3}{\sqrt{7+t} + 3\sqrt{9t-1}}$$

so that

$$f(t) = \frac{1}{(1-t)^4}\, h(t) \text{ for } t \geqslant \tfrac{1}{9}.$$

$h(t)$ is an increasing function of t, therefore

$$h(t) \geqslant h(\tfrac{1}{9}) = \frac{64}{9}. \quad \text{...(3.8)}$$

The following remarks are easy consequences of (3.3) and (3.8).

Remark 1 : $(1 - t)\, d \geqslant 1$, with equality if and only if $t = 0$ and $Q \sim Q_1$.

Remark 2 : $d \geqslant 1$, with equality if and only if $t = 0$ and $Q \sim Q_1$.

Lemma 6 — If $t = 0$ and $Q \sim Q_1$, then (3.7) holds. Equality is necessary if and only if $(x_0, y_0, z_0) \equiv (0, 0, 0) \pmod 1$.

PROOF : It is enough to consider the case $Q = Q_1$. If $y_0 \not\equiv 0 \pmod 1$, choose $y \equiv y_0 \pmod 1$ such that $0 < y < 1$, $x \equiv x_0 \pmod 1$ arbitrarily and then choose $z \equiv z_0 \pmod 1$ to satisfy

$$0 < x^2 + yz \leqslant y < 1$$

so that $td < x^2 + yz < d$.

If $z_0 \not\equiv 0 \pmod 1$, then the result is obtained by interchanging the choices of y and z. If $(y_0, z_0) \equiv (0, 0) \pmod 1$, take $y = z = 0$ and choose $x \equiv x_0 \pmod 1$ such that $0 < x \leqslant 1$, so that

$$td = 0 < x^2 + yz \leqslant x^2 \leqslant 1 = d.$$

Strict inequality holds unless $x_0 \equiv 0 \pmod 1$. When $(x_0, y_0, z_0) \equiv (0, 0, 0) \pmod 1$, equality holds in (3.7) as $x^2 + yz$ takes integral values only.

By Remark 1, we can now suppose that $(1 - t)d > 1$.

Lemma 7 — Let $Q(x, y, z)$ satisfy the conditions of Theorem A. Suppose that we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$- \nu_2 \leqslant \phi(y, z) < \nu_1 = d - \tfrac{1}{4} \qquad \ldots(3.9)$$

where

$$\nu_2 \begin{cases} \leqslant \left(\dfrac{(1 - t)\, d - 1}{2}\right)^2 - td & \text{if } (1 - t)d \text{ is an integer} \\ < \left(\dfrac{[(1 - t)d]}{2}\right)^2 - td & \text{if } (1 - t)d \text{ is not an integer.} \end{cases} \qquad \ldots(3.10)$$

Then there exists $x \equiv x_0 \pmod 1$ satisfying (3.7). Further strict inequality in (3.9) implies strict inequality in (3.7).

PROOF : If $td < \phi(y, z) < d - \tfrac{1}{4}$, choose $x \equiv x_0 \pmod 1$ such that

$$| x + hy + gz | \leqslant \tfrac{1}{2},$$

so that

$$td < Q(x, y, z) = (x + hy + gz)^2 + \phi(y, z) < d.$$

If $-\nu^2 \leqslant \phi(y, z) \leqslant td$, then the result follows from Lemma 2 with

$$\alpha = hy + gz,\ \beta^2 = td - \phi(y, z) \text{ and } r = (1 - t)\, d (> 1).$$

Divide the interval $0 \leqslant t < 1$ into disjoint subintervals

$$I_2 = [0, \tfrac{1}{9}]$$

$$I_n = \left(\left(\frac{n-2}{n}\right)^2, \left(\frac{n-1}{n+1}\right)^2\right], \qquad n = 3, 4, \ldots .$$

For $t \in I_n$, we consider the following cases:

(I) $(1 - t)\, d > n$

(II) $n - 1 < (1 - t)\, d \leqslant n$

(III) $n - 2 < (1 - t)\, d \leqslant n - 1, \quad n \geqslant 3$

(IV) $k < (1 - t)\, d \leqslant k + 1, \quad k = 1, 2, \ldots, n - 3, \quad n \geqslant 4.$

Lemma 8 — If $t \in I_n$ and $(1 - t)\, d > n\ (n \geqslant 2)$, then (3.7) holds with strict inequality.

PROOF : Since $(1 - t)\, d > n$ and $t \leqslant \left(\frac{n-1}{n+1}\right)^2$

$$\left(\frac{(1-t)\, d - 1}{2}\right)^2 - td > \left(\frac{n-1}{2}\right)^2 - t\,\frac{n}{1-t} \geqslant 0.$$

Let
$$\lambda = \frac{4d - 1}{\Delta} > 0.$$

By Lemma 3, we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\frac{\Delta}{4\lambda} \leqslant \phi(y, z) < \frac{\Delta\lambda}{4} = d - \tfrac{1}{4}.$$

The result will follow from Lemma 7, if we have

$$\frac{\Delta}{4\lambda} = \frac{d^3}{f(t)\,(4d - 1)} < \begin{cases} \left(\dfrac{(1-t)\, d - 1}{2}\right)^2 - td \text{ if } (1 - t)\, d \geqslant n + 1 \\ \dfrac{n^2}{4} - td \text{ if } n < (1 - t)\, d < n + 1. \end{cases} \qquad \ldots(3.11)$$

Case (*i*) : $(1 - t)\, d \geqslant n + 1$

(3.11) will be satisfied if

$$g(d) = \frac{\{(3 + t)\, d - 1\}^2}{4d^3} < (1 - t)^2 - \frac{1}{f(t)}.$$

$g(d)$ is a decreasing function of d and $d \geqslant \dfrac{n+1}{1-t}$, therefore

$$g(d) \leqslant g\left(\frac{n+1}{1-t}\right) = \frac{1-t}{4(n+1)^3}\{(3n+2)+t(n+2)\}^2.$$

This will be $< (1-t)^2 - \dfrac{1}{f(t)}$ if

$$f(t) > \frac{4(n+1)^3}{(1-t)(4n+3+t)(n^2-(n+2)^2 t)} = \delta(n, t) \text{ (say)}. \quad \ldots(3.12)$$

If $n = 2$, then $0 \leqslant t \leqslant \frac{1}{9}$ and (3.12) reduces to

$$f(t) = \frac{4}{(1-t)^2(1-5t)} > \delta(2, t) = \frac{27}{(11+t)(1-4t)(1-t)}.$$

If $n \geqslant 3$, then $t > \frac{1}{9}$ and (3.12) becomes

$$f(t) = \frac{1}{(1-t)^4} \cdot h(t) > \delta(n, t).$$

These can be easily verified to be true for $t \in I_n$. (We use $h(t) > 64/9$ for $t > \frac{1}{9}$).

Case (ii) : $n < (1-t)\,d < n+1$

In this case (3.11) is satisfied if

$$G(d) = \frac{d^3}{f(t)(4d-1)} + td < \frac{n^2}{4}. \quad \ldots(3.13)$$

By case (*i*)

$$G(d) < \left(\frac{(1-t)\,d-1}{2}\right)^2 \quad \text{for all } d \geqslant \frac{n+1}{1-t}.$$

In particular for $d = \dfrac{n+1}{1-t}$, we have

$$G\left(\frac{n+1}{1-t}\right) < \left(\frac{n+1-1}{2}\right)^2 = \frac{n^2}{4}.$$

Now (3.13) follows since $G(d)$ is an increasing function of d and $d < \dfrac{n+1}{1-t}$.

Lemma 9 — If $t \in I_n$, and $(n-1) < (1-t)\,d \leqslant n$, $n \geqslant 2$, then again (3.7) holds.

PROOF : In view of Lemma 7, it is enough to prove that there exist $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\left\{\left(\frac{n-1}{2}\right)^2 - td\right\} \leqslant \phi(y, z) < d - \tfrac{1}{4}. \qquad \ldots(3.14)$$

We notice that $\left(\frac{n-1}{2}\right)^2 - td > 0$, since $t \leqslant \left(\frac{n-1}{n+1}\right)^2$.

Let
$$\mu = \frac{(n-1)^2 - 4td}{4d-1}. \qquad \ldots(3.15)$$

Then $\mu \geqslant 0$. We shall have $\mu \leqslant \frac{1}{3}$ if $d \geqslant \frac{3(n-1)^2+1}{4(1+3t)}$.

Since $d > \frac{n-1}{1-t}$, this will be so if $t \geqslant \frac{3n^2-10n+8}{3n^2+6n-8}$. Now for $t \in I_n$, this is satisfied if $n \geqslant 4$ or $n = 2$. If $n = 3$, then $\mu \leqslant \frac{1}{3}$ if and only if $d \geqslant \frac{13}{4(1+3t)}$. So let us first suppose that $d \geqslant \frac{13}{4(1+3t)}$ when $n = 3$, and $\frac{n-1}{1-t} < d \leqslant \frac{n}{1-t}$ for all other n. Then $0 \leqslant \mu \leqslant \frac{1}{3}$ and by Lemma 4, there exist $(y, z) \equiv (y_0, z_0)$ (mod 1) such that

$$-\frac{\mu\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}} \leqslant \phi(y, z) < \frac{\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}}.$$

Then (3.14) will follow if

$$\frac{\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}} \leqslant d - \tfrac{1}{4}$$

i.e. if
$$(1+\mu)(1+9\mu) \geqslant \frac{\Delta^2}{(d-\frac{1}{4})^2} = \frac{64d^3}{f(t)(4d-1)^2}. \qquad \ldots(3.16)$$

Substituting for μ from (3.15) and simplifying we see that this is so if

$$g(d) = \frac{1}{d}\left\{\frac{(n-1)^2-1}{d} + 4(1-t)\right\} \times \left\{\frac{9(n-1)^2-1}{d} + 4(1-9t)\right\} \geqslant \frac{64}{f(t)}. \qquad \ldots(3.17)$$

All the factors of $g(d)$ are clearly decreasing functions of d and the first two are clearly positive. The third can also be verified to be positive for $d \leqslant \frac{n}{1-t}$ and $t \leqslant \left(\frac{n-1}{n+1}\right)^2$. Hence $g(d)$ is a strictly decreasing function of d. Therefore

$$g(d) \geqslant g\left(\frac{n}{1-t}\right)$$
$$= \frac{n+2}{n^2}(1-t)^2\{9n^2 - 14n + 8 - t(9n^2 + 18n + 8)\}.$$

Therefore (3.17) will be true if

$$f(t) \geqslant \frac{64n^2}{(n+2)(1-t)^2}\{(9n^2-14n+8)-t(9n^2+18n+8)\}^{-1} = f_n(t) \text{ (say)}.$$

For $n = 2, f_2(t) = f(t)$. For $n \geqslant 3, f(t) \geqslant \frac{64}{9}\frac{1}{(1-t)^4}$ and it is easy to verify that this is greater than $f_n(t)$.

Thus (3.17) is true with strict inequality unless $n = 2$.

It is clear from the proof that equality in (3.14) can occur only if

$$n = 2, d = \frac{n}{1-t} = \frac{2}{1-t} \text{ and equality in (2.5) holds.}$$

Now let us suppose that $n = 3$, and $\frac{2}{1-t} < d < \frac{13}{4(1+3t)}$. Let $\lambda = \frac{4d-1}{\Delta} (> 0)$.

By Lemma 3, there exist $(y, z) = (y_0, z_0) \pmod 1$ such that

$$-\frac{d^3}{f(t)(4d-1)} = -\frac{\Delta}{4\lambda} \leqslant \phi(y, z) < \frac{\lambda\Delta}{4} = d - \tfrac{1}{4}.$$

Then the result will follow from Lemma 7 if we have

$$\frac{d^3}{f(t)(4d-1)} < 1 - td$$

i.e. if $$g(d) = \frac{(4d-1)(1-td)}{d^3} > \frac{1}{f(t)}. \qquad \text{...(3.18)}$$

$g(d)$ is a decreasing function of d and $d < \frac{13}{4(1+3t)}$ therefore

$$g(d) \geqslant g\left(\frac{13}{4(1+3t)}\right) = \frac{16.3}{13^3}(4-t)^2(1+3t) \geqslant \frac{900}{13^3}$$

for $\frac{1}{9} < t \leqslant \frac{1}{4}$.

Also $\frac{1}{f(t)} < (1-t)^4 \frac{9}{64} \leqslant \frac{64}{9^3}$.

Thus (3.18) is satisfied and hence (3.14) holds with strict inequality.

Lemma 10 — If $t \in I_n$ and $n - 2 < (1-t)\, d \leqslant n - 1, n \geqslant 3$, then again (3.7) holds.

PROOF: In view of Lemma 7 it is enough to prove that there exist

$$(y, z) \equiv (y_0, z_0) \pmod 1$$

such that

$$-\left\{\frac{(n-2)^2}{4} - td\right\} \leqslant \phi(y, z) < d - \tfrac{1}{4}. \qquad \ldots(3.19)$$

Notice that $\frac{(n-2)^2}{4} - td \geqslant 0$ iff $d \leqslant \frac{(n-2)^2}{4t}$.

Consider the following cases:

Case (i) : $\frac{n-2}{1-t} < d \leqslant \frac{(n-2)^2}{4t}$.

Case (ii) : $\frac{(n-2)^2}{4t} < d \leqslant \frac{n-1}{1-t}$.

Proof of Case (i) — This case arises if and only if $t < \frac{n-2}{n+2}$.

Define $\mu = \frac{(n-2)^2 - 4td}{4d-1}$. Then $\mu \geqslant 0$. One can easily verify that $\mu \leqslant \frac{1}{3}$.

Now we use Lemma 4 and proceed as in the first part of the proof of Lemma 9. We see that (3.19) will follow if

$$g(d) = \frac{1}{d}\left\{\frac{(n-2)^2 - 1}{d} + 4(1-t)\right\}$$
$$\times \left\{\frac{9(n-2)^2 - 1}{d} + 4(1-9t)\right\} > \frac{64}{f(t)}. \qquad \ldots(3.20)$$

As before one can verify that $g(d)$ is a decreasing function of d. Therefore for $d \leqslant \frac{(n-2)^2}{4t}$

$$g(d) \geqslant g\left(\frac{(n-2)^2}{4t}\right) = \frac{64t((n-2)^2 - t)^2}{(n-2)^6}.$$

A straightforward calculation shows that this is greater than $64/f(t)$.

Proof of Case (ii) — When $\frac{(n-2)^2}{4t} < d \leqslant \frac{n-1}{1-t}$, we have

$$\frac{(n-2)^2}{4} - td < 0.$$

Let $0 \leqslant \nu < 1$ be a real number defined by

$$\frac{\Delta}{\{(1-\nu)^3(1+3\nu)\}^{1/2}} = d - \tfrac{1}{4} \qquad \ldots(3.21)$$

i.e. $\theta(\nu) = (1-\nu)^3(1+3\nu) - \frac{64d^3}{f(t)(4d-1)^2} = 0.$

Such a choice of ν is always possible as clearly $\theta(1) < 0$ and $\theta(0)$ can be verified to be positive for $d \leqslant \frac{n-1}{1-t}$, $t > \left(\frac{n-2}{n}\right)^2$, $n \geqslant 3$.

Therefore by Lemma 5, there exist $(y, z) = (y_0, z_0) \pmod 1$ satisfying

$$\frac{\nu^2 \Delta}{\{(1-\nu)^3(1+3\nu)\}^{1/2}} \leqslant \phi(y, z) < \frac{\Delta}{\{(1-\nu)^3(1+3\nu)\}^{1/2}}.$$

Then (3.19) will be true if we have

$$\frac{\nu^2 \Delta}{\{(1-\nu)^3(1+3\nu)\}^{1/2}} \geqslant td - \frac{(n-2)^2}{4}$$

i.e. $$\nu^2 \geqslant \frac{4td-(n-2)^2}{4d-1} = a^2 \text{ (say)}. \quad \ldots(3.22)$$

Since $\theta(\nu)$ is a decreasing function of ν to prove that $\nu^2 \geqslant a^2$ it suffices to prove $\theta(a) \geqslant \theta(\nu) = 0$

i.e. $$(1-a)^3(1+3a) \geqslant \frac{64d^3}{f(t)(4d-1)^2} \quad \ldots(3.23)$$

$$a^2 = \frac{4td-(n-2)^2}{4d-1} \text{ gives } d = \frac{(n-2)^2-a^2}{4(t-a^2)}.$$

Substituting for d in (3.23) and simplifying we see that $\theta(a) \geqslant 0$ if and only if

$$\psi(a) = \frac{(1-a)^3(1+3a)(t-a^2)}{((n-2)^2-a^2)^3} \geqslant \frac{1}{f(t)((n-2)^2-t)^2}. \quad \ldots(3.24)$$

A simple calculation shows that ψ is a decreasing function of a, and a^2, as a function of d is increasing. Therefore for $d \leqslant \frac{n-1}{1-t}$

$$a^2 \leqslant \frac{4t\frac{n-1}{1-t}-(n-2)^2}{4\frac{n-1}{1-t}-1} = \frac{n^2t-(n-2)^2}{4n-5+t}$$

and

$$\psi(a) \geqslant \psi\left(\sqrt{\frac{n^2t-(n-2)^2}{4n-5+t}}\right).$$

This will be $> \frac{1}{f(t)((n-2)^2-t)^2}$

if $$f(t) \geqslant \frac{4^3}{(n+1)^3}\frac{1}{(1-t)^4}\cdot\frac{(\sqrt{4n-5+t}+\sqrt{n^2t-(n-2)^2})^3}{(\sqrt{4n-5+t}+3\sqrt{n^2t-(n-2)^2})} \quad \ldots(3.25)$$

$$= f(n, t) \text{ (say)}.$$

One can easily verify that for a fixed $t \in I_n$, $n > 3$,

$$f(n, t) \leqslant f(3, t) = \frac{1}{(1-t)^4} \frac{(\sqrt{7+t} + \sqrt{9t-1})^3}{(\sqrt{7+t} + 3\sqrt{9t-1})} = f(t)$$

and the result follows.

It is clear from the proof that equality can occur in (3.19) only if we have equality in (2.6) and $n = 3$, $d = \frac{n-1}{1-t} = \frac{2}{1-t}$, $\nu^2 = a^2 = \frac{n^2 t - (n-2)^2}{4n - 5 + t} = \frac{9t-1}{7+t}$.

Lemma 11 — If $t \in I_n$ and $k < (1-t)\, d \leqslant k+1, k = 1, 2, \ldots, n-3, (n \geqslant 4)$, then again (3.7) is true.

PROOF : In view of Lemma 7, it is enough to prove that there exist

$$(y, z) = (y_0, z_0) \pmod 1$$

such that

$$td - \frac{k^2}{4} \leqslant \phi(y, z) < d - \tfrac{1}{4}. \qquad \ldots(3.26)$$

Since $d > \frac{k}{1-t}$ and $t > \left(\frac{n-2}{n}\right)^2$, we notice that $td - \frac{k^2}{4} > 0$.

Define a real number ν, $0 \leqslant \nu < 1$, such that

$$\frac{\Delta}{\{(1-\nu)^3 (1+3\nu)\}^{1/2}} = d - \tfrac{1}{4}.$$

Then working as in case (ii) of Lemma 10, we see that (3.26) is satisfied if $f(t) \geqslant f(k+2, t)$, where $f(n, t)$ is defined in (3.25). This is true as

$$f(k+2, t) < f(3, t) = f(t) \text{ for } k > 1.$$

Hence (3.26) is satisfied with strict inequality unless

$$k = 1, d = \frac{k+1}{1-t} = \frac{2}{1-t}, \nu^2 = a^2 = \frac{(k+2)^2 t - k^2}{4k + 3 - t} = \frac{9t-1}{7+t}.$$

By considering the cases of equality more closely, it is easy to verify that equality is required only for the special forms stated in the theorem. This completes the proof of Theorem A.

ACKNOWLEDGEMENTS

Authors are very grateful to Professor R. P. Bambah and Professor V. C. Dumir for many useful suggestions.

References

Blaney, H. (1950). Some asymmetric inequalities. *Proc. Camb. phil. Soc.*, **40**, 359–76.

Davenport, H. (1948). Non-homogeneous ternary quadratic forms. *Aeta Math.*, **80**, 65–95.

——— (1949). On indefinite ternary quadratic forms. *Proc. Lond. math. Soc.* (2), **51**, 145–60.

Dumir, V. C. (1968). Positive values of inhomogeneous quaternary quadratic forms (I). *J. Austr. math. Soc.*, **8**, 87–101.

Hans-Gill, R. J., and Madhu Raka (1980). Some inequalities for non-homogeneous quadratic forms I. *Indian J. pure appl. Math.*, **11**, 60–74.

Watson, G. L. (1960). Indefinite quadratic polynomials. *Mathematika*, 7, 141–44.