Indian J. pure appl. Math., 27(1) : 1-11, January 1996

SYMMETRIC STRUCTURES ON THE COTANGENT BUNDLES OF THE REAL AND COMPLEX GRASSMANNIANS

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(*Received* 1 *December* 1993; *after revision* 10 *March* 1995; *accepted* 30 *August* 1995)

The geometric structure of the cotangent bundles of the real oriented, real unoriented and complex Grassmann manifolds is studied. They are symmetric spaces in Berger's and Kaneyuki-Kozai's classifications. The cotangent bundle of the sphere and the real projective space are moreover harmonic symmetric spaces in Allamigeon's classification.

1. INTRODUCTION

As is well-known, the real and complex Grassmannians are important manifolds (see, for instance, the survey article by Borisenko and Nikolaevskii[4]). The tangent bundle of Grassmann manifolds has been studied, mainly from the point of view of Algebraic Topology, in Hsiang and Szczarba[15]; and the complex cotangent bundle of the complex Grassmannians in Calabi[5]. In the present paper we study some geometric structures on the cotangent bundles of the complex and — oriented or unoriented — real Grassmannians. They turn out to be symmetric spaces corresponding to symmetric pairs in : Berger's classification of semisimple symmetric pairs, Kaneyuki-Kozai's classification of semisimple para-Hermitian symmetric pairs and (in the case of the cotangent bundle of the sphere and the real projective space) Allamigeon's classification of semisimple harmonic symmetric pairs.

In the real 'projective' case, we have the spaces $P_n(B)$ and $P_n(B)/\mathbb{Z}_2$, introduced in Gadea and Amilibia[8, 10] which are endowed with a pseudo-Riemannian metric of 'Fubini-Study' type making them harmonic symmetric spaces[1, 9]. Instead of it, in the general case presently considered we have a pseudo-Riemannian 'Wong' metric (Wong[27] and §3 below), and the spaces have constant paraholomorphic sectional curvature only in the cases $P_n(B)$ and $P_n(B)/\mathbb{Z}_2$ (Gadea and Amilibia[8], Gadea and Masqué[13, 14]).

Since we obtain the explicit para-Kählerian structure of the above manifolds, we are giving specific examples in all — necessarily even — dimensions of para-Kählerian manifolds. This class of manifolds was independently defined in Rashevskij[26], Libermann[23] and Patterson[25] and it corresponds to a class in the classification given in Gadea and Masqué[12] of almost para-Hermitian manifolds (see Bejan[2]) which is the class with richest properties. Kaneyuki has given a large class of (homogeneous) para-Kählerian manifolds by means of semi-simple graded Lie algebras in Kaneyuki[16,17]. Other examples may be seen in references cited in this paper[2, 6, 8-14, 18-20]. The spaces introduced and classified in Kaneyuki and Kozai[18] are para-Hermitian symmetric and thus they are para-Kählerian spaces.

We also remark that the spaces under study are examples of quantizable coadjoint orbits (Kaneyuki and Williams[19]) in the sense of Kostant[22] and thus they are useful in theory of representations of the simple Lie groups $SL(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (Bien[3], Flensted-Jensen[7], Oshima and Matsuki[24]). Specifically, each of the spaces $G_{p,q}$ $(E \oplus E^*)_{\mathbb{R}}$, $G_{p,q}$ $(E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2$ and $G_{p,q}$ $(E \oplus E^*)_{\mathbb{C}}$ (see §2) is an open orbit of a spherical representation of the group $SL(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

In Gadea and Amilibia[11] the spaces G_q $(E \oplus E^*)$ and G_q $(E \oplus E^*)$ for a real vector space E have been introduced and their structure tensor fields have been defined, by means of suitable tensors in some spaces associated to the corresponding symmetric pair $(\mathfrak{sl}(n + 1, \mathbb{R}), \mathfrak{sl}(q, \mathbb{R}) + \mathfrak{sl}(n + 1 - q, \mathbb{R}) + \mathbb{R})$. This method, however, does not allow one to determine explicitly such tensor fields. Because of this, in the present paper we give a direct geometric definition of the aforementioned tensor fields, also including the complex case. To this end, we prove that principal bundles exist whose base manifolds are the para-Grassmannians, and that the metric tensor fields of the total spaces of such bundles project onto their corresponding base manifolds.

2. The Homogeneous Structure of the Cotangent Bundle of the Real and Complex Grassmannians

In what follows $\mathbb{F}$ stands either for $\mathbb{R}$ or for $\mathbb{C}$. Let E be a $(p + q)$-dimensional $\mathbb{F}$-vector space. On the space $E \oplus E^*$ we consider the following objects :

(1) A natural nondegenerate $\mathbb{F}$-bilinear form $\langle\ ,\ \rangle$ given by $\langle x+\alpha, y+\beta \rangle = \beta(x)+\alpha(y)$, where $x, y \in E, \alpha, \beta \in E^*$.

(2) An (1, 1) tensor J_0 defined by $J_0(x+\alpha) = x-\alpha$.

$GL(E)$ acts on $E \oplus E^*$ by the formula : $A \cdot (x + \alpha) = Ax + \alpha \circ A^{-1}$. Note that this action preserves both $\langle\ ,\ \rangle$ and J_0. We denote the identity component in $GL(E)$ by $GL_0(E)$.

Consider the Grassmann manifolds $G_p(E), G_p(E^*)$, $\tilde{G}_p(E), \tilde{G}_p(E^*)$ of p-planes in E, p-planes in E^*, oriented p-planes in E if $\mathbb{F} = \mathbb{R}$, and oriented p-planes in E^* if $\mathbb{F} = \mathbb{R}$, respectively.

Definition 2.1 — We shall call real para-Grassmannian, reduced real para-Grassmannian and complex para-Grassmannian, respectively, to the following spaces :

$G_{p,q}\ (E \oplus E^*)_{\mathbb{R}} = \{(F, \Lambda) \in \tilde{G}_p(E) \times \tilde{G}_p(E^*)$: the pairing $F \times \Lambda \to \mathbb{R}$, $(f, \lambda) \mapsto \lambda(f), f \in F, \lambda \in \Lambda$, is nondegenerate, and Λ has the induced orientation from that of F by means of the above pairing$\}$;

$G_{p,q}\ (E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2 = \{(F, \Lambda) \in G_p(E) \times G_p(E^*)$: the pairing $F \times \Lambda \to \mathbb{R}$, $(f, \lambda) \mapsto \lambda(f), f \in F, \lambda \in \Lambda$, is nondegenerate$\}$;

$G_{p,q}\ (E \oplus E^*)_{\mathbb{C}} = \{\{(F, \Lambda) \in G_p(E) \times G_p(E^*)$: the pairing $F \times \Lambda \to \mathbb{C}$, $(f, \lambda) \mapsto \lambda(f), f \in F, \lambda \in \Lambda$, is nondegenerate$\}$.

Remarks : (1) We have a nondegenerate symmetric $\mathbb{F}$-bilinear product $\langle\ ,\ \rangle$ on $F \oplus \Lambda$ given by $\langle f_1 + \lambda_1, f_2 + \lambda_2 \rangle = \lambda_1(f_2) + \lambda_2(f_1)$, and also an almost product automorphism J_0 given by $J_0(f + \lambda) = f - \lambda$. Thus, $F \oplus \Lambda$ is a natural substructure of $E \oplus E^*$. If $\mathbb{F} = \mathbb{R}$, the signature of the scalar product on $F \oplus \Lambda$ is (p, p).

(2) If $\mathbb{F} = \mathbb{R}$ and $q = 1$, then $G_{p,1}\ (E \oplus E^*)_{\mathbb{R}}$ and $G_{p,1}\ (E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2$ are, respectively, the spaces $P_p(B)$ and $P_p(B)/\mathbb{Z}_2$.

The following result is immediate :

Proposition 2.2 — The cotangent bundles of the oriented real, unoriented real and complex Grassmannians are diffeomorphic, respectively, to the real, reduced real and complex para-Grassmannians.

In view of the above result, we shall identify the cotangent bundles of the real or complex Grassmann manifolds to the (underlying C^ω real manifold of the) corresponding para-Grassmannian.

Consider now the open subset of $(E \oplus E^*)^p = E^p \oplus E^{*p}$ given by

$$(E \oplus E^*)^p_0 = \{x + \alpha = (x_1, ..., x_p; \alpha_1, ..., \alpha_p) \in (E \oplus E^*)^p\ :\ \det(\alpha_i(x_j)) \neq 0\}$$

and let

$$V_{p,q}(E \oplus E^*)_{\mathbb{F}} = \{(x_1, ..., x_p; \alpha_1, ..., \alpha_p) \in (E \oplus E^*)^p : \alpha_i(x_j) = \delta_{ij}\},$$

which is a submanifold in $(E \oplus E^*)^p_0$. Then, the group $GL_0(E)$ acts transitively on $V_{p,q}(E \oplus E^*)_{\mathbb{F}}$ as one can easily prove. Hence $(V_{p,q}(E \oplus E^*)_{\mathbb{F}}, GL_0(E))$ is a homogeneous manifold.

Let $x^0 + \alpha^0 = (x^0_1, ..., x^0_p;\ \alpha^0_1, ..., \alpha^0_p) \in V_{p,q}(E \oplus E^*)_{\mathbb{F}}$. Consider $x^0_{p+1}, ..., x^0_{p+q}$ and $\alpha^0_{p+1}, ..., \alpha^0_{p+q}$ such that (x^0_i) and (α^0_i), $1 \le i \le p + q$, are bases of E and E^* dual of each other. Then, the isotropy group of $x^0 + \alpha^0$ can be identified with $GL_0(q, \mathbb{F})$. Consequently,

$$V_{p,q}(E \oplus E^*)_{\mathbb{F}} = GL_0(p + q, \mathbb{F})/GL_0(q, \mathbb{F}) = SL(p + q, \mathbb{F})/SL(q, \mathbb{F}).$$

Moreover, we have a principal bundle $\pi : V_{p,q}(E \oplus E^*)_{\mathbb{F}} \to G_{p,q}(E \oplus E^*)_{\mathbb{F}}$, with projection map $\pi(x_1, ..., x_p; \alpha_1, ..., \alpha_p) = (\langle x_1, ..., x_p \rangle;\ \langle \alpha_1, ..., \alpha_p \rangle)$. The structure group is $GL_0(p, \mathbb{F})$ acting—on the right—as follows : $(x, \alpha) \cdot S = (x \cdot S, \alpha \cdot S)$, $(x \cdot S)_a =$

$\Sigma_b s_{ba} x_b$, $(\alpha \cdot S)_a = \Sigma_b s^{ab} \alpha_b$, where $x = (x_1, ..., x_p)$, $\alpha = (\alpha_1, ..., \alpha_p)$, $S = (s_{ab}) \in GL_0(p, \mathbb{F})$ and (s^{ab}) stands for the inverse matrix of $S = (s_{ab})$.

By passing to the quotient, we also obtain a principal bundle $\bar{\pi} : V_{p,q}(E \oplus E^*)_{\mathbb{R}} \to G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbf{Z}_2$ having structural group $GL(p, \mathbb{R})$. It is easy to see, by considering the isotropy group of the element $\pi(x^0 + \alpha^0)$, that

$$G_{p,q}(E \oplus E^*)_{\mathbb{F}} \approx GL_0(p+q, \mathbb{F})/GL_0(p, \mathbb{F}) \times GL_0(q, \mathbb{F})$$

$$\approx SL(p+q, \mathbb{F})/S(GL_0(p, \mathbb{F}) \times GL_0(q, \mathbb{F})) \quad ... (2.1)$$

and

$$G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbf{Z}_2 \approx GL_0(p+q, \mathbb{R})/(GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))_0$$

$$\approx SL(p+q, \mathbb{R})/S(GL(p, \mathbb{R}) \times GL(q, \mathbb{R})). \quad ... (2.2)$$

Thus $\dim_{\mathbb{F}} G_{p,q}(E \oplus E^*)_{\mathbb{F}} = 2pq$, $\dim_{\mathbb{R}} G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbf{Z}_2 = 2pq$, and we have that $G_{p,q}(E \oplus E^*)_{\mathbb{R}}$ is a twofold covering manifold of $G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbf{Z}_2$.

Proposition 2.3 — The cotangent bundles of the oriented real or complex, and unoriented real Grassmannians are homogeneous spaces, respectively diffeomorphic to the homogeneous spaces (2.1) and (2.2).

3. The Para-Kähler Structure of the Cotangent Bundle of the Real and Complex Grassmannians

We shall now endow the real manifold $(G_{p,q}(E \oplus E^*)_{\mathbb{F}})_{\mathbb{R}}$ underlying $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ with an almost para-Hermitian structure; that is, a pseudo-Riemannian metric $g_{\mathbb{R}}$ and an almost product structure $J_{\mathbb{R}}$ compatible with $g_{\mathbb{R}}$ as an anti-isometry.

From now on, we shall adopt the following convention for indices :

$$1 \le a, b, c, d \le p; \ 1 \le u, v \le q; \ 1 \le i, j, k, l \le p+q.$$

Let (e_i) be a basis of E with dual basis (ω_i). For every $x + \alpha \in (E \oplus E^*)^p$ we set $\omega_{ia}(x) = \omega_i(x_a)$, $e_{ia}(\alpha) = e_i(\alpha_a)$. We can consider (ω_{ia}, e_{ia}) as a system of coordinates in $(E \oplus E^*)^p$. For each subset $\{i_1, ..., i_p\}$ of $\{1, ..., p+q\}$ with $1 \le i_1 < ... < i_p \le p+q$, let $U_{i_1,...,i_p}$ be the open subset of $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ defined as follows :

$$U_{i_1,...,i_p} = \{\pi(x+\alpha) \in G_{p,q}(E \oplus E^*)_{\mathbb{R}} :$$
$$\det(\omega_{ia}(x)) > 0, \ \det(e_{ia}(\alpha)) > 0, \ i \in \{i_1, ..., i_p\}\},$$

$$U_{i_1,...,i_p} = \{\pi(x+\alpha) \in G_{p,q}(E \oplus E^*)_{\mathbb{C}} :$$
$$\det(\omega_{ia}(x)) \ne 0, \ \det(e_{ia}(\alpha)) \ne 0, \ i \in \{i_1, ..., i_p\}\}.$$

We define $2pq$ coordinates (x_a^u, y_a^u) on $U_{i_1,...,i_p}$ by setting

$$x_a^u(\pi(x+\alpha)) = \omega_{p+u,b}(x)\,\omega^{ba}(x), \ y_a^u(\pi(x+\alpha)) = e_{p+u,b}\,e^{ba}(\alpha), \quad ... (3.1)$$

where $(\omega^{ab}(x))$ and $(e^{ab}(\alpha))$ denote the inverse matrices of $(\omega_{ab}(x))$ and $(e_{ab}(\alpha))$, respectively. Then, $G_{p,q}\ (E \oplus E^*)_{\mathbb{F}}$ is an open submanifold either of $\tilde{G}_p(E)_{\mathbb{R}} \times \tilde{G}_p(E^*)_{\mathbb{R}}$ or $G_p(E)_{\mathbb{C}} \times G_p(E^*)_{\mathbb{C}}$, and an atlas is given by

$$\{(U_{i_1,\dots,i_p}, \phi_{i_1,\dots,i_p}) : \phi_{i_1,\dots,i_p}(\pi(x+\alpha)) = (x_a^u(\pi(x+\alpha)),\ y_a^u(\pi(x+\alpha)))\}. \quad \dots (3.2)$$

Similar considerations are also valid for the case $G_{p,q}\ (E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2$, taking the condition $\det(\omega_{ab}(x)) \cdot \det(e_{ab}(\alpha)) > 0$ in the definition of the coordinate domains $U_{i_1,\dots,i_p}$.

We now endow the real manifold $(G_{p,q}\ (E \oplus E^*)_{\mathbb{F}})_{\mathbb{R}}$ underlying $G_{p,q}\ (E \oplus E^*)_{\mathbb{F}}$ with a $GL_0(E)$-invariant pseudo-Riemannian metric. First, we note that the space $(E \oplus E^*)^p$ has a nondegenerate symmetric $\mathbb{F}$-bilinear form $\langle\ ,\ \rangle$ and a (1, 1) tensor J_0 which extend the ones given in $E \oplus E^*$, namely, $\langle x+\alpha, y+\beta\rangle = \alpha(y) + \beta(x)$, $x, y \in E^p$, $\alpha, \beta \in E^{*p}$, $\alpha(x) = \Sigma_{a=1}^p\ \alpha_a(x_a)$, $J_0(x+\alpha) = x - \alpha$.

Moreover, the group $GL_0(E)$ acts on $(E \oplus E^*)^p$ by the standard representation. Also note that the group $GL(p, \mathbb{F})$ acting on the right on $(E \oplus E^*)^p$ as explained above (i.e., $(x \cdot S)_a = \Sigma_b\, s_{ba}\, x_b$, $(\alpha \cdot S)_a = \Sigma_b\, s^{ab}\, \alpha_b$) is a group of isometries of the bilinear form $\langle\ ,\ \rangle$, as it is easily checked. Since $GL_0(E)$ acts upon $(E \oplus E^*)^p$ by isometries, and preserves $V_{p,q}(E \oplus E^*)_{\mathbb{F}}$, it also acts by isometries on $V_{p,q}(E \oplus E^*)_{\mathbb{F}}$. Then, given $Y \in T_{\pi(x+\alpha)}\, G_{p,q}(E \oplus E^*)_{\mathbb{F}}$, we can lift Y to a vector $Y^H \in T_{x+\alpha}\, V_{p,q}(E \oplus E^*)_{\mathbb{F}}$ orthogonal to the fibre such that $\pi_*(Y^H) = Y$. This process has a unique solution if the restriction of the metric $\langle\ ,\ \rangle$ on $(E \oplus E^*)_0^p$ to the tangent space $T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha)))$ at $x+\alpha$ to the fibre of $\pi : V_{p,q}(E \oplus E^*)_{\mathbb{F}} \to G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ is nondegenerate in $T_{x+\alpha}(E \oplus E^*)_0^p$. Actually, let $\mu : GL(p, \mathbb{F}) \to (E \oplus E^*)^p$ be the map $\mu(S) = (x+\alpha) \cdot S$ where $S = (s_{ab})$. Note that $\mu_*(T_I(GL(p, \mathbb{F}))) = T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha))$, and, as a computation shows, we have

$$\mu_*\left(\frac{\partial}{\partial s_{ab}}\bigg|_I\right) = \omega_{ia}(x)\frac{\partial}{\partial \omega_{ib}}\bigg|_{x+\alpha} - e_{ib}(\alpha)\frac{\partial}{\partial e_{ia}}\bigg|_{x+\alpha} \quad \dots (3.3)$$

Hence, $\langle \mu_*(\partial/\partial s_{ab})|_I),\ \mu_*(\partial/\partial s_{cd})|_I)\rangle = -\delta_{ad}\,\delta_{bc}$, thus proving that $T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha)))$ is nonsingular. Since $\langle\ ,\ \rangle$ is nondegenerate in $(E \oplus E^*)_0^p$, it follows that

$$T_{x+\alpha}(E \oplus E^*)_0^p = T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha)) \oplus (T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha))))^\perp,$$

where $(T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha))))^\perp$ is the orthogonal subspace of $T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha)))$, in $T_{x+\alpha}(E \oplus E^*)_0^p$, and consequently,

$$T_{x+\alpha}\ (V_{p,q}(E \oplus E^*)_{\mathbb{F}}) = T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha))) \oplus (T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha))))_V^\perp.$$

This allows us to define Y^H along $\pi^{-1}(\pi(x+\alpha))$ for every $Y \in T_{\pi(x+\alpha)}\ G_{p,q}(E \oplus E^*)_{\mathbb{F}}$.

as explained above. Since $T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha)))$ is nonsingular and the right translation by S, R_S, is an isometry, we have $Y^H_{(x+\alpha)\cdot S} = (R_s)_*(Y^H_{x+\alpha})$. This property leads us to a well-defined metric g on $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ by setting

$$g(Y,Z) = \langle Y^H_{x+\alpha}, Z^H_{x+\alpha} \rangle, \; Y, Z \in T_{\pi(x+\alpha)} G_{p,q}(E \oplus E^*)_{\mathbb{F}}. \quad \text{... (3.4)}$$

In order to prove that g is nondegenerate it will suffice to prove that the space $T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha)))^{\perp}_V$ is a nonsingular subspace with respect to the metric $\langle\, , \rangle$. To do so, in turn, it will suffice to prove that $T_{x+\alpha}(V_{p,q}(E \oplus E^*)_{\mathbb{F}})^{\perp}$ is nonsingular. Let $f : (E \oplus E^*)^p \to \mathbb{F}^{p^2}$ be the map whose components are given by $f_{ab}(x+\alpha) = \alpha_a(x_b)$. Hence, $V_{p,q}(E \oplus E^*)_{\mathbb{F}} = f^{-1}(I)$. Consequently, $T_{x+\alpha}(V_{p,q}(E \oplus E^*)_{\mathbb{F}})^{\perp} = \oplus^p_{a,b=1} \langle (\operatorname{grad} f_{ab})_{x+\alpha} \rangle_{\mathbb{F}}$. Then, as a calculation shows,

$$(\operatorname{grad} f_{ab})_{x+\alpha} = \omega_{ib}(x) \left.\frac{\partial}{\partial \omega_{ia}}\right|_{x+\alpha} + e_{ia}(\alpha) \left.\frac{\partial}{\partial e_{ib}}\right|_{x+\alpha}. \quad \text{... (3.5)}$$

Hence, $\langle (\operatorname{grad} f_{ab})_{x+\alpha}, (\operatorname{grad} f_{cd})_{x+\alpha} \rangle = 2\delta_{ad}\delta_{bc}$, thus finishing the proof.

Furthermore, $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ admits an $\mathbb{F}$-almost product structure J which we now define. The group $G = GL_0(p, \mathbb{F}) \times GL_0(p, \mathbb{F})$ acts on $(E \oplus E^*)^p_0$ by $(x+\alpha)(S,T) = (x \cdot S, \alpha \cdot T)$, where $(x \cdot S)_a = \Sigma_b s_{ba} x_b$, $(\alpha \cdot T)_a = \Sigma_b t^{ab} \alpha_b$, S, $T \in G$, and $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ is the quotient space of that action.

Let J_0 be the almost product structure with respect to $\mathbb{F}$ on $(E \oplus E^*)^p_0$ defined by $J_0(v, \theta) = (v, -\theta), (v, \theta) \in T_{x+\alpha}(E \oplus E^*)^p_0$, where we have used the natural identification of $T_{x+\alpha}(E \oplus E^*)^p_0$ with the vector space $(E \oplus E^*)^p$. Accordingly, $(J_0 \circ R_{(A,B)*} - R_{(A,B)*} \circ J_0)(v, \theta) = J_0(Av, \theta \circ B^{-1}) - (A,B)_*(v, -\theta) = 0$. Hence, the action passes to the quotient, giving the $\mathbb{F}$-almost product structure J, which is $GL_0(E)$-invariant since so J_0 is. Moreover, it is immediate from the definitions of $\langle\, , \rangle$, g and J that g is $\mathbb{F}$-para-Hermitian, in the sense that $g(JX, Y) + g(X, JY) = 0$, $X, Y \in \chi(G_{p,q}(E \oplus E^*)_{\mathbb{F}})$, and it is also clear that the eigenbundles of J are g-isotropic.

The corresponding structure on $G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2$ is constructed in a similar way. Also note that the almost complex para-Hermitian structure (g, J) on $G_{p,q}(E \oplus E^*)_{\mathbb{C}}$ induces a real almost para-Hermitian structure $(g_{\mathbb{R}}, J_{\mathbb{R}})$ on the real underlying manifold to $G_{p,q}(E \oplus E^*)_{\mathbb{C}}$ by setting

$$g_{\mathbb{R}} = \operatorname{Re}\, g \text{ and } J_{\mathbb{R}}(X) = \operatorname{Re}\, J(X - iI(X)), \quad \text{... (3.6)}$$

for every $X \in T((G_{p,q}(E \oplus E^*)_{\mathbb{C}})_{\mathbb{R}})$, I being the almost complex structure associated with the complex manifold $G_{p,q}(E \oplus E^*)_{\mathbb{C}}$.

We shall prove that $((G_{p,q}(E \oplus E^*)_{\mathbb{F}})_{\mathbb{R}}, g_{\mathbb{R}}, J_{\mathbb{R}})$ is in fact a para-Kähler manifold. For this, consider the coordinate neighbourhood $U_{1,\dots,p}$ (see (3.2)) of $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$, and let (x^u_a, y^u_a) be the coordinates on $U_{1,\dots,p}$. From now on we shall sometimes drop the $x+\alpha$ or $\pi(x+\alpha)$ in the vectors which will appear. Recall that $x_a^u = \omega_{p+u,b}\,\omega^{ba}$, $y_a^u = e_{p+u,b}\,e^{ba}$. Then, as a computation shows, $\partial x_a^u/\partial\omega_{ib} = \omega^{ba}(\delta_{p+u,i} - \delta_{ic}\,x_c^u)$, from which one obtains

$$\left.\frac{\partial}{\partial x_a^u}\right|_{\pi(x+\alpha)} = \pi_{*x+\alpha}\left(\omega_{ab}\left.\frac{\partial}{\partial\omega_{p+u,b}}\right|_{x+\alpha}\right), \qquad \text{... (3.7)}$$

where $\pi : (E\oplus E^*)_0^p \to G_{p,q}(E\oplus E^*)_{\mathbf{F}}$ is the natural projection.

A given tangent vector $X\in T_{x+\alpha}\,(E\oplus E^*)_0^p$ can be uniquely written as

$$X = X_F + X_F^{\perp} + X_V^{\perp}, \qquad \text{... (3.8)}$$

where the vectors $X_F\in T_{x+\alpha}(\pi^{-1}(\pi(x+\alpha)))$, $X_F^{\perp}\in T_{x+\alpha}(\pi^{-1}((\pi(x+\alpha)))_V^{\perp}$ and $X_V^{\perp}\in T_{x+\alpha}\,(V_{p,q}\,(E\oplus E^*)_{\mathbf{F}})^{\perp}$ are pairwise $\langle\,,\rangle$-orthogonal vectors. Then $\pi_*(X) = \pi_*\,(X_F^{\perp})$. If $X = \omega_{ab}/(\partial/\partial\omega_{p+u,b})$, then from (3.3) and (3.5) we know that

$$X_F = \lambda_{bc}^{ua}\left(\omega_{ib}\frac{\partial}{\partial\omega_{ic}} - e_{ic}\frac{\partial}{\partial e_{ib}}\right) \qquad \text{... (3.9)}$$

and

$$X_V^{\perp} = \mu_{bc}^{ua}\left(\omega_{ib}\frac{\partial}{\partial\omega_{ic}} + e_{ic}\frac{\partial}{\partial e_{ib}}\right) \qquad \text{... (3.10)}$$

for certain λ_{bc}^{ua}, μ_{bc}^{ua}, which we can explicitly determine by means of a calculation. In account of the fact that $x+\alpha\in V_{p,q}\,(E\oplus E^*)_{\mathbf{F}}$ we deduce

$$\lambda_{bc}^{ua} = \mu_{bc}^{ua} = \frac{1}{2}\,\omega_{ac}\,e_{p+u,b}. \qquad \text{... (3.11)}$$

Similar computations hold for $(\partial/\partial y_a^u)\,|_{\pi(x+\alpha)}$. Hence, according to (3.7)-(3.11) we obtain

$$\left(\frac{\partial}{\partial x_a^u}\right)^H = \omega_{ab}\left(\frac{\partial}{\partial\omega_{p+u,b}} - \omega_{ic}\,e_{p+u,c}\frac{\partial}{\partial\omega_{ib}}\right)$$

and

$$\left(\frac{\partial}{\partial y_a^u}\right)^H = e_{ab}\left(\frac{\partial}{\partial e_{p+u,b}} - e_{ic}\,\omega_{p+u,c}\frac{\partial}{\partial e_{ib}}\right).$$

Consequently, from (3.4) we deduce

$$g(\partial/\partial x_a^u, \partial/\partial x_b^v) = \langle(\partial/\partial x_a^u)^H, (\partial/\partial x_b^v)^H\rangle = 0,\quad g(\partial/\partial y_a^u, \partial/\partial y_b^v) = 0,$$

and

$$g(\partial/\partial x_a^u, \partial/\partial y_b^v) = \delta_{uv}\,\omega_{ac}\,e_{bc} - \omega_{ac}\,\omega_{p+v,d}\,e_{bc}\,e_{p+u,d}. \qquad \text{... (3.12)}$$

Consider the matrices $\mathbf{X} = (x_a^u)$, $\mathbf{Y} = (y_a^u)$. From (3.1), we have $x_a^u = \omega_{p+u,b}\,\omega^{ba}$, $y_a^u = e_{p+u,b}\,e^{ba}$. If we denote $\Omega = (\omega_{ia}), E = (e_{ia})$, we have

$$\Omega = \begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix}, \quad E = \begin{pmatrix} E_0 \\ E_1 \end{pmatrix},$$

where Ω_0 and E_0 are $(p \times p)$-matrices. On $V_{p,q}(E \oplus E^*)_{\mathbb{F}}$ one has ${}^t\Omega E = I_p$; that is, ${}^t\Omega_0 E_0 + {}^t\Omega_1 E_1 = I_p$. Letting $\Omega_1 = \mathbf{X}\,\Omega_0$, $E_1 = \mathbf{Y}\,E_0$, we deduce

$$I_p + {}^tYX = {}^tE_0^{-1}\,\Omega_0^{-1} = (\Omega_0\,{}^tE_0)^{-1}. \qquad \text{... (3.13)}$$

Substituting (3.12) in the right-hand side of (3.13), it follows

$$g = (I_p + {}^t\,\mathbf{YX})^{-1}_{ab}\,(\delta_{uv} - x_c^v y_d^u\,(I_p + {}^t\,\mathbf{YX})^{-1}_{cd})\,(dx_a^u \otimes dy_b^v + dy_b^v \otimes dx_a^u). \qquad \text{... (3.14)}$$

Consider the matrices $d\mathbf{X} = (dx_a^u)$, $d\mathbf{Y} = (dy_a^u)$, and let $dx_a^u \cdot dy_b^v = dx_a^u \otimes dy_b^v + dy_b^v \otimes dx_a^u$. According to (3.6), the C^ω manifold underlying $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ has the metric $g_{\mathbf{R}}$ = Re g; that is, from (3.14) we obtain

$$g_{\mathbf{R}} = \operatorname{Re}\operatorname{Tr}\,[(I + {}^t\,\mathbf{YX})^{-1}\,\{d^t\,\mathbf{Y} \cdot d\mathbf{X} - d^t\,\mathbf{Y} \cdot \mathbf{X} \cdot (I + {}^t\,\mathbf{YX})^{-1} \cdot {}^t\,\mathbf{Y} \cdot d\mathbf{X}\}]. \qquad \text{... (3.15)}$$

As for $J_{\mathbf{R}}$, we can take $J\left(\frac{\partial}{\partial x_a^u}\Big|_{\pi(x+\alpha)}\right) = \pi_* J_0\left(\frac{\partial}{\partial x_a^u}\Big|^H_{\pi(x+\alpha)}\right) = \pi_*\left(\frac{\partial}{\partial x_a^u}\Big|^H_{\pi(x+\alpha)}\right) = \frac{\partial}{\partial x_a^u}\Big|_{\pi(x+\alpha)}$ and, similarly, $J\left(\frac{\partial}{\partial y_a^u}\Big|_{\pi(x+\alpha)}\right) = -\frac{\partial}{\partial y_a^u}\Big|_{\pi(x+\alpha)}$ That is, $J = \partial/\partial x_a^u \otimes dx_a^u - \partial/\partial y_a^u \otimes dy_a^u$ and

$$J_{\mathbf{R}}\,(X) = \operatorname{Re}\left(\left(\frac{\partial}{\partial x_a^u} \otimes dx_a^u - \frac{\partial}{\partial y_a^u} \otimes dy_a^u\right)(X - iI(X))\right). \qquad \text{... (3.16)}$$

Let $\mathbb{F} = \mathbb{C}$. Then the tangent space $T_{\pi(x+\alpha)}\,G_{p,q}(E \oplus E^*)_{\mathbb{C}}$ at $\pi(x+\alpha)$ admits the basis $\{(\partial/\partial x_a^u)\,|_{\pi(x+\alpha)}, (\partial/\partial y_a^u)\,|_{\pi(x+\alpha)}\}$. Let $x_a^u = x_{a1}^u + ix_{a2}^u$, $y_a^u = y_{a1}^u + iy_{a2}^u$ on a neighbourhood of $\pi(x+\alpha)$. Then the tangent space $(T_{\pi(x+\alpha)}\,G_{p,q}(E \oplus E^*)_{\mathbb{C}})_{\mathbf{R}}$ admits the basis $\{\partial/\partial x_{a1}^u, \partial/\partial x_{a2}^u, \partial/\partial y_{a1}^u, \partial/\partial y_{a2}^u\}$, and the complex almost product structure J induces on $(G_{p,q}(E \oplus E^*)_{\mathbb{C}})_{\mathbf{R}}$ the almost product structure $J_{\mathbf{R}}$, which in the same neighbourhood has the expression

$$J_{\mathbf{R}} = \frac{\partial}{\partial x_{a1}^u} \otimes dx_{a1}^u + \frac{\partial}{\partial x_{a2}^u} \otimes dx_{a2}^u - \frac{\partial}{\partial y_{a1}^u} \otimes dy_{a1}^u - \frac{\partial}{\partial y_{a2}^u} \otimes dy_{a2}^u.$$

In account of equation (3.6), the 2-form$F_{\mathbf{R}}$ is defined by $F_{\mathbf{R}}(X, Y) = g_{\mathbf{R}}(J_{\mathbf{R}}\,X, Y)$ = (Re g) $(J_{\mathbf{R}}\,X, Y)$. Since g and J are $GL_0(p+q, \mathbb{F})$-invariant, F and dF are also $GL_0(p+q, \mathbb{F})$-invariant. Thus, in order to compute dF it suffices to know its value at $\mathbf{X} = \mathbf{Y} = 0$. But, as a computation shows, $dF\,|_0 = 0$. Hence $dF = 0$, and $d(F_{\mathbf{R}}) = (dF)_{\mathbf{R}} = 0$.

Furthermore, since obviously $N = 0$, N being the Nijenhuis tensor of $J_{\mathbb{R}}$, it follows from [Gadea and Amilibia[8], Cor. 3.2, p. 87] that $(G_{p,q}(E \oplus E^*)_{\mathbb{C}})_{\mathbb{R}}$ is a para-Kähler manifold. The same result is immediate if $\mathbb{F} = \mathbb{R}$, and also we get it in the case $G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2$, since $G_{p,q}(E \oplus E^*)_{\mathbb{R}}$ is a para-Hermitian two-fold covering manifold of $G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2$. We have thus proved :

Proposition 3.1 — The cotangent bundles of the real and complex Grassmannians, endowed with the pseudo-Riemannian Wong's metric and the almost product structure expressed in the coordinates (3.1) respectively by (3.15) and (3.16), are para-Kähler manifolds.

4. The Symmetric Structure of the Cotangent Bundle of the Real and Complex Grassmannians

Now, we shall prove that the spaces $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ and $G_{p,q}(E \oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2$ are normal symmetric spaces. Actually, let us consider the involutive automorphism σ of the group $G = GL_0(p+q, \mathbb{F})$ given by

$$\sigma(A) = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} A \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad A \in GL_0(p+q, \mathbb{F}).$$

The subgroup of elements invariant by σ is $G^\sigma = (GL(p, \mathbb{F}) \times GL(q, \mathbb{F}))_0$, and the identity component of G^σ is $G_0^\sigma = GL_0(p, \mathbb{F}) \times GL_0(q, \mathbb{F})$. Thus the isotropy group H of G is G_0^σ in the case $G_{p,q}(E \oplus E^*)_{\mathbb{F}}$ and G^σ in the case $G_{p,q}(E \oplus E^*)_{\mathbb{F}}/\mathbb{Z}_2$.

Moreover, let $\mathfrak{g} = \mathfrak{gl}(p+q, \mathbb{F})$ be the Lie algebra of G. Then, the inner product $\langle , \rangle$ given by $\langle X, Y \rangle = \mathrm{Re}\,\mathrm{Tr}\,(XY)$, $X, Y \in \mathfrak{g}$ is Ad(G)-invariant, and we have $\langle d\sigma(X), d\sigma(Y) \rangle = \langle X, Y \rangle$. One has

$$\begin{aligned} \mathfrak{m} &= \{X \in \mathfrak{g} : d\sigma(X) = -X\} \\ &= \left\{ \begin{pmatrix} 0 & A \\ V & 0 \end{pmatrix} : V \in \mathbb{F}^{pq},\ A \in \mathbb{F}^{*pq} \right\} \approx \mathbb{F}^{pq} \oplus \mathbb{F}^{*pq}. \end{aligned}$$

Let J_0 be the almost product automorphism of $\mathfrak{m}$ defined by

$$J_0 X = J_0 \begin{pmatrix} 0 & A \\ V & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A \\ V & 0 \end{pmatrix}, \quad X \in \mathfrak{m},$$

and let $g_0 = \langle , \rangle|\mathfrak{m}$. Then

$$g_0(X, Y) = \mathrm{Re}\,\mathrm{Tr} \begin{pmatrix} 0 & A \\ V & 0 \end{pmatrix} \begin{pmatrix} 0 & A' \\ V' & 0 \end{pmatrix} = \mathrm{Re}\,\mathrm{Tr}\,(V' \otimes A + V \otimes A'),$$

from which $g_0(J_0 X, Y) + g_0(X, J_0 Y) = 0$, $X, Y \in \mathfrak{m}$.

It is immediate that g_0 and J_0 are Ad(H)-invariant. By means of the usual identifications (Kobayashi and Nomizu[21], Vol. II, Ch. XI) $\mathbb{F}^{pq} \oplus \mathbb{F}^{*pq}$

$\approx m \xrightarrow{d\pi} T_0(G/H)$, we obtain that the extensions via left translations of g_0 and J_0 are the fields (3.15) and (3.16) on G/H, since these fields are G-invariant and $d\pi$ is a paraholomorphic isometry. Consequently, corresponding to the symmetric Lie algebra $(s\ell(p+q, \mathbb{C}), s\ell(p, \mathbb{C}) + s\ell(q, \mathbb{C}) + \mathbb{C})$, we have the para-Hermitian symmetric space given by (2.1) for $\mathbb{F} = \mathbb{C}$. Similarly, corresponding to the symmetric Lie algebra $(s\ell(p+q, \mathbb{R}), s\ell(p, \mathbb{R}) + s\ell(q, \mathbb{R}) + \mathbb{R})$, we have the para-Hermitian symmetric spaces given by (2.1) for $\mathbb{F} = \mathbb{R}$ and (2.2). They are the spaces corresponding to the first and third series in Kaneyuki-Kozai's classification[18].

It is moreover immediate from Allamigeon's infinitesimal classification[1] of harmonic symmetric spaces, that in the case $q = 1$, $\mathbb{F} = \mathbb{R}$ (and only in this case) the spaces are harmonic symmetric. They are the spaces $P_p(B)$ and $P_p(B)/\mathbb{Z}_2$, which are the cotangent bundles, respectively, of the sphere S^p and the projective space $P_p(\mathbb{R})$. We can thus state :

Proposition 4.1 — The cotangent bundles of the real and complex Grassmannians, are the symmetric spaces corresponding to the symmetric pairs

$$(s\ell(p+q, \mathbb{F}), s\ell(p, \mathbb{F}) + s\ell(q, \mathbb{F}) + \mathbb{F}), \quad \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C},$$

in :

(a) Berger's infinitesimal classification of semisimple symmetric pairs;

(b) Kaneyuki-Kozai's infinitesimal classification of semisimple para-Hermitian symmetric pairs (when endowed with the para-Kähler structure given by the metric $g_{\mathbb{R}}$ and the almost product structure $J_{\mathbb{R}}$ displayed in §3);

(c) Allamigeon's infinitesimal classification of semisimple harmonic symmetric pairs (when $\mathbb{F} = \mathbb{R}$ and $q = 1$).

The cotangent bundles of the real or (resp. complex) Grassmannians are para-Hodge (resp. nontrivial para-Hodge) manifolds in the sense of Kaneyuki and Williams[20]; that is, the real cohomology class $[F_{\mathbb{R}}]$ of the fundamental 2-form $F_{\mathbb{R}}$ is an integer cohomology class.

The para-Hodge structure is induced by the Killing form of the corresponding Lie algebra $s\ell(p+q, \mathbb{R})$ (resp. $s\ell(p+q, \mathbb{C})$). We can add :

Corollary 4.2 — Kostant's 2-form of the cotangent bundles of the real and complex Grassmannians, endowed with the structures displayed in §3 is given, with the above notations, by

$$F_{\mathbb{R}} = \mathrm{Re}\ \mathrm{Tr}\ [(I + {}^t\mathbf{Y}\mathbf{X})^{-1}\ \{d^t\mathbf{Y} \wedge d\mathbf{X} - d^t\mathbf{Y} \cdot \mathbf{X} \wedge (I + {}^t\mathbf{Y}\mathbf{X})^{-1} \cdot {}^t\mathbf{Y} \cdot d\mathbf{X}\}].$$

PROOF : The result follows from Kaneyuki and Williams[20], (3.15) and (3.16).

ACKNOWLEDGEMENT

We are grateful to Prof. A. Montesinos Amilibia for his valuable suggestions.

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