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EFFECTS OF DIFFERENTIAL ROTATION ON THE PERIODS OF SMALL ADIABATIC OSCILLATIONS OF STARS IN BINARY SYSTEMS

V. P. SINGH AND M. K. SHARMA

University of Roorkee, Institute of Paper Technology, Saharanpur 247 001

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In this paper we present a method for computing the eigenfrequencies of small adiabatic oscillations of stellar models distorted by differential rotation and tidal forces. The method is based on the approach adopted by Mohan and Singh[23] in conjunction with the averaging concept introduced by Kippenhahn and Thomas[17]. The angular velocity of rotation is assumed to be the function of the square of the distance of fluid element from the axis of rotation. Tidal distortions are assumed to be caused by a nearby point mass. Such studies have practical importance in astrophysics in determining the periods of small adiabatic oscillations of differentially rotating stars in binary systems. Comparison of results with observational data is also presented.

1. INTRODUCTION

In an earlier paper (Singh and Sharma[27], hereafter shall be referred to as paper I) we have presented a method to compute the periods of small adiabatic oscillations of stellar models distorted by differential rotation. Observation however, shows that certain variable stars rotate about their axes and some of them have also been observed in binary systems (see for instance Fitch[7], Jarzebowski *et al.*[15], Fitch and Wisniewski[8]). In the case of such type of variable stars if small adiabatic pulsations are considered to be the primary cause of stellar variability, then the observed periods of pulsations of such stars are being influenced by the rotational effects in case of rotating stars and rotational and tidal effects in the case of the variable stars in binary system. Observations indicate that many of the rotating stars do not have solid body rotation, but are rotating differentially. In the case of differentially rotating stars in binary system, it is natural to expect that the rotation and tidal forces distort their otherwise spherically symmetric structures. However, mathematical problem of determining the effects of rotation and tidal forces on the eigenfrequencies of the possible modes of small adiabatic oscillations of realistic models of stars in binary system is quite complex. Approximate methods have, therefore, been used in literature to study such problems. In one such approximation, the inner structure of a star is approximated by a Roche model and investigations are made on it to get an insight into the nature of the inner-structure, oscillations and stability of the realistic models of stars.

In fact, whereas the problem of determining the eigen frequencies of the radial and nonradial modes of oscillations of undistorted models of the stars has been extensively studied in literature (for detailed reference see Ledoux and Walraven[19], Cox[4]), Saio[28] has tried to discuss the effects of rotation and tidal distortion on the eigenfrequencies of nonradial modes of oscillations of polytropic models. Martins and Smeyers[20] investigated the problem of linear adiabatic oscillations of a uniformly and synchronously rotating component of a binary system. Kopal[16] introduced the concept of Roche equipotentials to study the problems of close binary stars. Kippenhahn and Thomas[17] proposed a method for determining the structure, oscillations and stability of rotationally and tidally distorted stellar models. Since then several investigators such as Mohan and Singh[21, 22], Mohan and Saxena[24], Mohan *et al.*[25], Chan and Chau[5], Tassoul and Tassoul[31], Todaran[33], Iben[13], Mohan *et al.*[26] have discussed the structure and oscillation of a rotating model of the primary component of a close binary system. The analysis developed by Mohan and Singh[23] for studying the effects of rotation and tidal distortion on the modes of oscillations of stars is applicable only to the stars having solid body rotation. In the present paper we investigate the problem of determining the eigenfrequencies of small adiabatic oscillations of the Roche model of a star distorted by differential rotation and tidal force. In the present paper, we use the averaging concept of Kippenhahn and Thomas[17] in conjunction with Kopal's[16] concept of Roche equipotential for studying the problems of rotationally and tidally distorted stellar models.

In section 2, we present some of the laws of differential rotation which have been used earlier in literature for the analysis of differentially rotating stars. The explicit expressions of r_ξ, V_ξ, S_ξ, $\bar{g}$, and $\overline{g^{-1}}$, for differentially rotating and tidally distorted stellar models are given in section 3. The eigenvalue problem of determining the periods of small adiabatic oscillations of differentially rotating Roche model of star obeying the law $w = b_1 + b_2 s^2$ in close binary system is then presented in section 4. Evaluation of numerical results are next discussed in section 5 and depicted in Table II. Some conclusion and comparison with observations based on this study are presented in section 6.

2. Laws of Differential Rotation

By differential rotation we mean rotation of a gaseous sphere in which all the fluid elements of the sphere do not have the same angular velocity. Some of the authors such as Von Zeipal[34], Solberg[29], Hoiland[11], etc, used a law of differential rotation for star rotating about an axis of rotation passing through its centre, of the type $\Omega = \Omega(s, \phi)$ in which angular velocity Ω is a function of both distance s from the axis of rotation in units of the radius of the star, and the latitude ϕ. However, according to Tassoul[32], it is perhaps not possible to build a chemically homogeneous stellar model in radiative equilibrium with a rotation law of the type $\Omega = \Omega(s, \phi)$. According to him, in the zones of efficient convection the transport of energy is not by radiation, so in such a case Von Zeipal's argument does not apply. According to him, therefore, in principle, for such a differentially rotating star, a law of differential rotation of the form $\Omega = \Omega(s)$ may well be used. Ireland[14] used such type of law in a rapidly rotating Roche model of a star subject to nonuniform rotation assuming $\Omega = \Omega(s)$ where Ω is the angular velocity of the star and s is the distance of a fluid element from the axis of rotation. Stoeckly[30] have used a law of differential rotation

$$\Omega(s) = \Omega_C \, e^{-(as/\xi_e^2)}$$

where Ω_c is the angular velocity on the axis of rotation, ξ_e is the equatiorial radius of the polytropic model and a is a suitably chosen constant. Bodenheimer[3] used a law which gives the angular momentum per unit mass $J(m)$ as a function of m, the mass interior to a given cylinder about the axis of rotation. Geroyannis *et al.*[9] obtained a complete solution of structural equation for differentially rotating polytropes by taking a differential rotation law which is a function of position and time dependent homoaxial rotation. Geroyannis and Antonakopoulos[10] used a law of differential rotation earlier proposed by Clement[6], according to this law the angular velocity $w(s)$ of a fluid element is given by

$$w(s) = \left(\sum_{i=1}^{3} a_i \, e^{-b_i s^2} \right)^{1/2}$$

where s is a modified nondimensional cylindrical coordinate and a_i, b_i are some constants. Komatsu *et al.*[18] calculated equilibrium structure of differentially relativistic polytropes with indices 0.5 and 1.5 using a rotation law determined by specifying the angular momentum $J(\Omega)$.

$$J(\Omega) = A^2 (\Omega_C - \Omega)$$

where A is positive constant and Ω_c is the angular velocity at the centre of the coordinate system, (Ω_c depends implicitly on the value of A which is called rotation parameter). For the Newtonian case this leads to the rotation law of the type

$$\Omega = \frac{\Omega_C A^2}{A^2 + s^2}$$

where $s = r \sin \Theta$, Θ being the co latitude. When $A \to \infty$, Ω approaches a rigid rotation. When $A \to 0$, it becomes a J-constant rotation. Woodard considered a law of differential rotation of the type

$$\Omega(\phi) = B_0 + B_1 \phi^2 + B_2 \phi^4$$

where Ω is even function of latitude ϕ.

In our present study we have used a law of differential rotation (as suggested by FAYE as early as 1865 for the Sun's surface as it appears to fit the observations quite well, at least to high solar latitudes) of the type

$$w = b_1 + b_2 s^2$$

where w is the angular velocity of fluid element distant s ($s = r \sin \Theta$ is a nondimensional distance i.e. s and r expressed in units of the radius of the star) from the axis of rotation passing through its centre and b_1, b_2 are suitably chosen arbitrary constants. Ireland[14] has also used this law for calculating the effects of differential rotation on the gravity-darkening and limb-darkening for Roche model under going the differential rotation. Ireland indicated that besides this law fit to the observation quite well, this law may find useful application to early type stars.

Following Ireland[14] we have preferred to use the above type of law of differential rotation because we are also studying the effects of differential rotation and tidal distortion on the Roche model of stars. This law of differential rotation not only generates a variety of differential rotations commonly expected in stars but it is also in a form in which it can be conveniently subjected to the type of mathematical analysis which we propose to carry out in the present study. In this law of differential rotation there is symmetry of angular velocity about the axis of rotation. Angular velocity varies with s and is the same on the surface of a cylinder of radius s whose axis is the same as the axis of rotation. Behaviour of the angular velocity in the equatorial plane from the axis of rotation to the equator is reproduced in the behaviour of the angular velocity on the surface of the star from the pole to equator. For a suitable choice of b_1 and b_2, the angular velocity can be made to increase as well as decrease from the axis of rotation outwards (or equivalently, from the pole towards the equator on the surface). We can also generate the situations in which on moving outwards from the axis of rotation towards the surface (or equivalently, from the pole towards equator on the surface), the angular velocity partly decreases and partly increases or there is partly solid body rotation (or no rotation) and partly differential rotation.

Following Kopal[16] a binary stellar system is assumed to have two components known as primary and secondary stars. Of these two, the primary star is supposed to be more massive than the secondary which acts as a point mass causing tidal effects on the more massive primary component. The primary massive star is assumed to follow Roche model as its inner structure. Both the component of binary system are assumed to be rotating about their axis as well as revolving about their common centre of gravity. In our present investigation the primary star is assumed to have differential instead of solid body rotation. The secondary star being much smaller than the primary, only makes its contribution causing tidal effects on the primary which is of more interest to us in our present study. For a model rotating differentially according to the law $w = b_1 + b_2 s^2$ to be stable it has to satisfy Stoeckly criteria (Stoeckly[30]), which requires that

$$\frac{d}{ds}\,[s^2\, w(s)] > 0$$

for all values of s inside the star. The nature of certain types of differential rotation which can be generated by $w = b_1 + b_2 s^2$ by giving different values to b_1 and b_2 are shown in Table I.

TABLE I

Behaviour of angular velocity in certain differentially rotating model

Model No.	Values of various parameters in the law of differential rotation $w = b_1 + b_2 s^2$		Behaviour of square of the angular velocity w^2 from axis of rotation ($s = 0$) to equator ($s = 1$) in the equatorial plane (from pole to equator on the surface for a differentially rotating model in which s is in the units of equatorial radius R_e	Stability of the model according to Stoeckly criterion
	b_1	b_2		
1	0.0	0.0	Non-rotating model	stable
2	0.1	0.0	solid body rotation about axis of rotation in which w^2 is .01 throughout the model	stable
3	0.1	0.1	w^2 increases gradually from .01 to .04	stable
4	0.1	0.02	w^2 increases first slowly then rapidly from .01 to .014	stable
5	0.1	0.03	w^2 increases slowly form .01 to .016	stable
6	0.1	0.04	w^2 increases more rapidly from .01 to .019	stable
7	0.1	− 0.05	w^2 decreases gradually from .01 to .002	stable
8	0.0	− 0.05	w^2 increases first slowly and then more rapidly from 0.0 to .002	unstable
9	0.0	0.02	w^2 increases first slowly then rapidly from 0.0 to .0004	stable
10	0.0	0.03	w^2 increases first slowly then rapidly from 0.0 to .0008	stable

3. EXPLICIT EXPRESSIONS OF r_ξ, V_ξ, s_ξ, $\bar{g}$, AND $\bar{g}^{-1}$, FOR DIFFERENTIALLY ROTATING AND TIDALLY DISTORTED STELLAR MODELS

Suppose M_0 and M_1 are the masses of the two components of a close binary system separated by a distance R. We further suppose that the primary component of this system of mass M_0 is much larger than its companion star of mass M_1 which can be regarded as a point mass. We also suppose that the undistorted equilibrium structure of the primary star is a Roche model which is rotating about its axis. The total potential Ω of the gravitational and disturbing force acting at an arbitrary point $P(X, Y, Z)$ is given by

$$\Omega = G\frac{M_0}{r*} + G\frac{M_1}{r_1^*} + \frac{1}{2}w^2\left[\left(x - \frac{M_1R}{M_0+M_1}\right)^2 + y^2\right] \quad \text{... (1)}$$

where G is the constant of gravitation, r^* and $r_1{}^*$ respectively are the distance of the point P from the centres of masses M_0 and M_1 and w the angular velocity of rotation of the system. Here the origin has been chosen at the centre of the mass M_0, X-axis along the line joining the centres of two masses and Z-axis along the axis of rotation. In nondimensional form eqn. (1) can be expressed as

$$\xi = \frac{1}{r} + q\sum_{j=0}^{\infty}[P_j(\lambda)\,r^j - \lambda r] + \frac{w^2}{2}r^{*2}(1-\nu^2) \quad \text{... (2)}$$

where ξ is the nondimensional form of the total potential, $r = r^*/R$, $\lambda = \sin\Theta\cos\phi$, $\mu = \sin\Theta\sin\phi$, $\nu = \cos\Theta$ (r, Θ, ϕ being the polar spherical coordinates of the point P). $P_j(\lambda)$ are the Legendre polynomials. Also $q = M_0/M_1$ and w is the nondimensional parameter denoting the angular velocity. In eqn. (2) if we set $q = 0$, it reduces to the potential of a spherical configuration rotating with angular velocity w and if we set $w = 0$, we obtained the potential of a spherical configuration distorted by the tidal effects of a companion. For a binary system in synchronous rotation w and q are connected through the relation

$$w^2 = q + 1. \quad \text{... (3)}$$

In the case of differentially rotating model the angular velocity of rotation in the model changes from point to point. Using the proposed law of differential rotation equation (2) may be written as

$$\xi = \frac{1}{r} + q\sum_{j=0}^{\infty}[P_j(\lambda)\,r^j - \lambda r]$$
$$+\frac{1}{2}r^2(1-\nu^2)\left[b_1^2 + b_1b_2\,r^2(1-\nu^2) + \frac{1}{3}b_2^2\,r^4(1-\nu^2)^2\right] \quad \text{... (4)}$$

as the nondimensional form at a point P inside a differentially rotating gaseous sphere, whose equilibrium structure is approximated by a Roche model and which is subject to the tidal influence of a neighbouring mass. Following the approach adopted by Mohan and Singh[23], it can be shown that (r, Θ, ϕ) on such equipotential surface are connected through the relation

$$r = r_0\left[1 + c_3r_0^3 + c_4r_0^4 + c_5r_0^5 + c_6r_0^6 + c_7r_0^7 + c_8r_0^8 + c_9r_0^9 + c_{10}r_0^{10} + ...\right] \quad \text{... (5)}$$

where

$$c_3 = qp_2 + \frac{1}{2}b_1^2(1-\nu^2), \qquad c_4 = qp_3,$$

$$c_5 = qp_4 + \frac{1}{2} b_1 b_2 (1 - \nu^2)^2 ,$$

$$c_6 = qp_5 + 3q^2 p_2^2 ,$$

$$c_7 = qp_6 + 7q^2 p_2 p_3 + \frac{1}{6} b_2^2 (1 - \nu^2)^3$$

$$c_8 = qp_7 + 8q^2 p_2 p_4 + 4q^2 p_3^2,$$

$$c_9 = qp_8 + 9q^2 p_2 p_5 + 9q^2 p_3 p_4,$$

$$c_{10} = qp_9 + 10q^2 p_2 p_6 + 10q^2 p_3 p_5 + 5q^2 p_4^2,$$

where $r_0 = 1/(\xi - q)$, and terms up to second order of smallness in b_1, b_2 and q are retained. Following the same approach, explicit expressions for V_ξ, S_ξ, r_ξ, $\bar{g}$, and $\bar{g}^{-1}$ can obtained. These are

$$V_\xi = \frac{4}{3} \pi r_0^3 \left[1 + b_1^2 r_0^3 + \frac{4}{5} b_1 b_2 r_0^5 + \frac{12}{5} q^2 r_0^6 + \frac{8}{35} b_2^2 r_0^7 + \frac{16}{7} q^2 r_0^8 + 2q^2 r_0^{10} + \ldots\ldots \right] \quad \ldots (6)$$

$$S_\xi = 4\pi r_0^2 \left[1 + \frac{2}{3} b_1^2 r_0^3 + \frac{8}{15} b_1 b_2 r_0^5 + \frac{7}{5} q^2 r_0^6 + \frac{16}{105} b_2^2 r_0^7 + \frac{9}{7} q^2 r_0^8 + \frac{11}{9} q^2 r_0^{10} + \ldots\ldots \right] \quad \ldots (7)$$

$$r_\xi = r_0 \left[1 + \frac{1}{3} b_1^2 r_0^3 + \frac{4}{15} b_1 b_2 r_0^5 + \frac{4}{5} q^2 r_0^6 + \frac{8}{105} b_2^2 r_0^7 + \frac{16}{21} q^2 r_0^8 + \frac{2}{3} q^2 r_0^{10} + \ldots\ldots \right] \quad \ldots (8)$$

$$\bar{g} = \frac{1}{r_0^2} \left[4 - \frac{4}{3} b_1^2 r_0^3 - \frac{24}{15} b_1 b_2 r_0^5 - 2 q^2 r_0^6 - \frac{64}{105} b_2^2 r_0^7 - \frac{15}{7} q^2 r_0^8 - \frac{21}{9} q^2 r_0^{10} - \ldots\ldots \right] \quad \ldots (9)$$

$$\bar{g}^{-1} = r_0^2 \left[1 + \frac{4}{3} b_1^2 r_0^3 + \frac{24}{15} b_1 b_2 r_0^5 + \frac{26}{5} q^2 r_0^6 + \frac{64}{105} b_2^2 r_0^7 + \frac{40}{7} q^2 r_0^8 + \frac{57}{9} q^2 r_0^{10} - \ldots\ldots \right] \quad \ldots (10)$$

4. Eigenvalue Problem Determining the Effects of Small Adiabatic Radial Modes of Oscillations of Differentially Rotating Stars in Close Binary Systems

The problem of determining the effects of differentially rotating stars in binary systems is quite complex. Mohan and Singh[23] used the concept of Kippenhahn and Thomas to formulate an eigenvalued boundary value problem to determine the periods of small adiabatic radial modes of oscillations of rotationally and tidally distorted Roche model. The approach adopted by Mohan and Singh[23] can also be used to set up the eigenvalue problem which determines the periods of small adiabatic radial modes of oscillations of differentially rotating stars in close binary systems.

For the Roche model, density distribution ρ is given by $\rho = \varepsilon r^{-\alpha}$, where α is some positive real number. Also the equipotential surface ξ is constant. We can suppose that the density $\rho(\xi)$ on the surface of topologically equivalent sphere of radius r_ξ is given by

$$\rho(\xi) = \varepsilon\, r_\xi^{-\alpha}. \qquad \text{... (11)}$$

Following Kippenhahn and Thomas[17], we can write

$$\frac{dM_\xi}{dr_\xi} = 4\pi r_\xi^2\, \rho(\xi)$$

using eqn. (11), we can have

$$\frac{dM_\xi}{dr_\xi} = 4\pi\, \varepsilon\, r_\xi^{2-\alpha}$$

so that

$$M_\xi = M + 4\pi\varepsilon \frac{r_\xi^{3-\alpha}}{(3-\alpha)} \qquad \text{... (12)}$$

where M is the mass of discrete central mass point. The total mass M' of the Roche model is given by

$$M' = \frac{4\pi}{3} R_\xi^3\, \bar{\rho} + M. \qquad \text{... (13)}$$

Provided we choose

$$M = \frac{4}{3}\pi R_\xi^3\, \bar{\rho} \left[1 - \frac{3\varepsilon}{\bar{\rho}\,(3-\alpha)\, R_\xi^\alpha} \right].$$

Again for hydrostatic equilibrium

$$\frac{dP(\xi)}{d\xi} = -\rho(\xi)$$

so that
$$\frac{dP(\xi)}{dr_\xi} = -\rho(\xi) \frac{4\pi\, r_\xi^2}{g^{-1} S_\xi}.$$

Thus

$$\frac{1}{\rho(\xi)}\frac{dP(\xi)}{dr_\xi} = -\frac{4\pi r_\xi^2}{\bar{g}^{-1} S_\xi}. \qquad \text{... (14)}$$

Taking r_0 as the independent variable and making some simplification eqn. (14) can be written as

$$\frac{dP(\xi)}{dr_0} = -\varepsilon\left[r_0^{-4} - \frac{2}{3}b_1^2 r_0^{-1} - \frac{18}{15}b_1 b_2 r_0 - \frac{18}{5}q^2 r_0^2 - \frac{16}{105}b_2^2 r_0^3 - \frac{1}{7}q^2 r_0^4 - \frac{2}{9}q^2 r_0^6 - \dots\right].$$

On integration this leads to

$$P(\xi) = \frac{\varepsilon U}{3r_0^3}\left[1 + \frac{3b_1^2 \log(r_0) r_0^3}{U} + \frac{4}{5U}b_1 b_2 r_0^5 + \frac{18}{5U}q^2 r_0^6 + \frac{92}{105}b_2^2 r_0^7 + \frac{3}{35}q^2 r_0^8 + \frac{2}{210}q^2 r_0^{10} + \dots\right]. \qquad \text{... (15)}$$

Here

$$U = 1 + 3Dr_0^3$$

and

$$D = -\frac{1}{3R_0^3}\left[1 + 2b_1^2 R_0^3 \log(R_0) + \frac{4}{5}b_1 b_2 R_0^5 + \frac{18}{5}q^2 R_0^6 + \frac{12}{105}b_2^2 R_0^7 + \frac{3}{35}q^2 R_0^8 + \frac{2}{21}q^2 R_0^{10} + \dots\right].$$

Here terms up to second order of smallness in b_1, b_2 and q are retained and for r_0 it is up to r_0^{10}.

Now

$$\mu = -\frac{r_\xi}{P(\xi)}\frac{dP(\xi)}{dr_\xi}$$

Then using the values of r_ξ, ρ_ξ, $P(\xi)$, and μ in the equation

$$\frac{d^2\varsigma}{dr_\xi^2} + \frac{4-\mu}{r_\xi}\frac{d\varsigma}{dr_\xi} + \left[\frac{w^{*2}}{\Gamma P(\xi)} - \left(3 - \frac{4}{\Gamma}\right)\frac{\mu}{r_\xi^2}\right]\varsigma = 0. \qquad \text{... (16)}$$

We may write the pulsation eqn. (16) for a Roche model distorted by the combined effects of differential rotation and tidal forces are

$$A(r_0)\frac{d^2\varsigma}{dr_0^2} + B(r_0)\frac{d\varsigma}{dr_0} + \left[\frac{w^{*2}}{\Gamma}D(r_0) - \left(3 - \frac{4}{\Gamma}\right)E(r_0)\right]\varsigma = 0 \qquad \text{... (17)}$$

where

$$A(r_0) = 1 - \frac{8}{3} b_1^2 r_0^3 - \frac{16}{5} b_1 b_2 r_0^5 - \frac{56}{5} q^2 r_0^6 - \frac{128}{105} b_2^2 r_0^7 - \frac{96}{7} q^2 r_0^8 - \frac{44}{3} q^2 r_0^{10} - \ldots \qquad \ldots (18)$$

$$B(r_0) = \frac{1}{r_0}\left[4 - \frac{3}{U} + \left\{ \frac{32}{3} + \frac{10}{U} + \frac{6\log(r_0)}{U^2} \right\} b_1^2 r_0^3 + \left\{ -\frac{232}{15} + \frac{56}{5U} + \frac{12}{5U^2} \right\} b_1 b_2 r_0^5 + \left\{ -\frac{296}{5} + \frac{222}{5U} + \frac{54}{5U^2} \right\} q^2 r_0^6 + \left\{ -\frac{736}{105} + \frac{144}{35U} + \frac{12}{35U^2} \right\} b_2^2 r_0^7 + \left\{ -\frac{1792}{21} + \frac{291}{7U} + \frac{9}{35U^2} \right\} q^2 r_0^8 + \left\{ -\frac{316}{3} + \frac{134}{3U} + \frac{2}{7U^2} \right\} q^2 r_0^{10} + \ldots \right] \qquad \ldots (19)$$

$$D(r_0) = r_0 \left[\frac{3}{U} - \left\{ \frac{2}{U} - \frac{6\log(r_0)}{U^2} \right\} b_1^2 r_0^3 - \left\{ \frac{8}{5U} + \frac{12}{5U^2} \right\} b_1 b_2 r_0^5 - \left\{ \frac{24}{5U} + \frac{54}{5U^2} \right\} q^2 r_0^6 - \left\{ \frac{16}{35U} + \frac{2}{35U^2} \right\} b_2^2 r_0^7 - \left\{ \frac{32}{7U} + \frac{9}{35U^2} \right\} q^2 r_0^8 - \left\{ \frac{4}{U} + \frac{12}{7U^2} \right\} q^2 r_0^{10} + \ldots \right] \qquad \ldots (20)$$

$$E(r_0) = \frac{1}{r_0^2} \left[\frac{3}{U} - \left\{ \frac{7}{U} + \frac{6\log(r_0)}{U^2} \right\} b_1^2 r_0^3 - \left\{ \frac{36}{5U} + \frac{12}{5U^2} \right\} b_1 b_2 r_0^5 - \left\{ \frac{30}{U} + \frac{54}{5U^2} \right\} q^2 r_0^6 - \left\{ \frac{88}{35U} + \frac{12}{35U^2} \right\} b_2^2 r_0^7 - \left\{ \frac{163}{7U} + \frac{9}{35U^2} \right\} q^2 r_0^8 - \left\{ \frac{22}{U} + \frac{2}{7U^2} \right\} q^2 r_0^{10} + \ldots \right] \qquad \ldots (21)$$

w^{*2} is the nondimensional form of frequency of oscillations.

In case of no distortion $b_1 = b_2 = q = 0$ and $r_\xi = r_0 = r$. Therefore $R_0 = R$, the radius of star now becomes the unit of the distance in the definition of the nondimensional potential ξ. Hence $r_0 = r$ is now the usual dimensionless variable $x = r/R$, defined in the case of the problem of stellar structure. In case of no distortion therefore,

$$r_0 = x, \quad R_0 = 1, \quad r_\xi = r_0, \quad \mu = 3/(1 - x^3),$$

so that

$$A(r_0) = 1, \quad B(r_0) = \frac{1}{x}\left(4 - \frac{3}{(1 - x^3)} \right)$$

$$D(r_0) = \frac{3x}{\Gamma(1-x^3)}, \quad E(r_0) = 3(3-4/\Gamma)/x^2\,(1-x^3).$$

Using these results we notice that in case of no distortion the eqn. (17) of pulsation reduces to

$$(1-x^3)\frac{d^2 Ç}{dx^2} + \frac{(1-4x^3)}{x}\frac{dÇ}{dx} + 3\left[\frac{x}{\Gamma}w^2 - \left(3-\frac{4}{\Gamma}\right)\frac{1}{x^2}\right]Ç = 0. \qquad \text{... (22)}$$

This is the usual well known equation of small adiabatic oscillations of the undistorted Roche model. If we put $q = 0$, eqn. (17) reduces to the pulsation equation having only differential rotation effect as we obtained in an earlier paper (1995). If we set $b_1 = b_2 = 0$, eqn. (17) becomes the pulsation equation for Roche model having tidal forces effects only.

5. Numerical Evaluation of the Results

Equation (17) forms an eigenvalue problem in the square of the nondimensional frequency of oscillations w^*. This eigenvalue problem is of Sturm-Liouville type and has to be solved to boundary condition which require Ç to be finite at the centre and free surface of the model. Usual techniques of numerical integration used for solving such types of problems for the case of undistorted models can also be used in this case.

To determine the eigen values of the frequencies of small oscillations, series solution were developed near the singular points where as a hypergeometric series solution of the type given by Kopal[16] is still valid near the centre. A series of the type

$$Ç = 1 + \sum_{k=0}^{\infty} c_k \left(1 - \frac{r_0}{R_0}\right)^k \qquad \text{... (23)}$$

is developed near the surface, (this obviously implies that we are assuming Ç to be normalized to have value one at $r_0 = R_0$). To determine an eigen value, a trial value of w^{*2} is chosen. For this value of w^{*2} series (23) is used to determine the values of Ç at points near R_0. Integration is then carried inwards towards centre numerically with the help of eqn. (17). At the points, near the centre, the values of Ç are obtained from hypergeometric series solution which is valid near the centre. Trial with different values of w^{*2} is continued till a value of w^{*2} is found for which the values of Ç obtained from inward integration agreed to a desired accuracy with the corresponding value of Ç obtained at these points from hypergeometric series solution. Computation have been performed for certain values of b_1, b_2 and q etc. The results are presented in Table II.

TABLE II

Eigen values of fundamental mode and first mode of oscillations for a Roche model distorted by differential rotation and tidal force

ξ_1	b_1	b_2	q	Γ	Eigen values of w^{*2} of	
					Fundamental Mode	First Mode
0.4	0.0	0.00	0.00	1.6667	5.7494	15.9128
0.4	0.0	0.00	0.00	1.5000	4.6342	13.7288
0.4	0.1	0.10	0.01	1.6667	4.5612	10.4655
0.4	0.1	0.10	0.01	1.5000	3.6067	8.9695
0.4	0.1	0.02	0.01	1.6667	3.2011	9.0336
0.4	0.1	0.02	0.01	1.5000	2.7381	7.7162
0.4	0.1	0.03	0.01	1.6667	3.3736	8.7703
0.4	0.1	0.03	0.01	1.5000	2.7561	7.4985
0.4	0.1	0.04	0.01	1.6667	3.3068	8.2720
0.4	0.1	0.04	0.01	1.5000	2.6351	7.0781
0.4	0.1	– 0.05	0.01	1.6667	3.8236	8.8593
0.4	0.1	– 0.05	0.01	1.5000	3.0733	7.6296
0.4	0.0	– 0.05	0.01	1.6667	3.0242	7.5820
0.4	0.0	– 0.05	0.01	1.5000	2.3732	6.4628
0.5	0.0	0.00	0.00	1.6667	5.3224	13.5319
0.5	0.0	0.00	0.00	1.5000	4.5000	13.6881
0.5	0.0	0.00	0.01	1.6667	5.1567	19.1764
0.5	0.0	0.00	0.01	1.5000	4.3037	16.6058
0.5	0.0	0.00	0.02	1.6667	4.8068	12.8748
0.5	0.0	0.00	0.02	1.5000	3.7126	10.8184
0.5	0.1	0.10	0.00	1.6667	4.7410	16.2818
0.5	0.1	0.10	0.00	1.5000	3.7670	14.0171
0.5	0.1	0.20	0.00	1.6667	4.6780	11.0467
0.5	0.1	0.20	0.00	1.5000	3.5792	10.3576
0.5	0.1	– 0.05	0.00	1.6667	2.0439	5.7077
0.5	0.1	– 0.05	0.00	1.5000	1.5571	4.9408

(Continued on page 115)

Table II — Continued

ξ_1	b_1	b_2	q	Γ	Eigen values of w^{*2} of	
					Fundamental Mode	First Mode
0.5	0.1	0.10	0.01	1.6667	4.7888	11.8339
0.5	0.1	0.10	0.01	1.5000	3.8140	9.9046
0.5	0.1	0.10	0.02	1.6667	5.2491	12.3074
0.5	0.1	0.10	0.02	1.5000	4.0943	10.4986
0.6	0.0	0.00	0.00	1.6667	3.2705	9.4432
0.6	0.0	0.00	0.00	1.5000	2.3373	7.8977
0.6	0.1	0.10	0.02	1.6667	4.4029	12.1550
0.6	0.1	0.10	0.02	1.5000	3.3311	10.0372
0.6	0.1	0.10	0.03	1.6667	4.3133	11.6013
0.6	0.1	0.10	0.03	1.5000	3.1805	9.6788
0.6	0.0	0.02	0.01	1.6667	3.0238	8.3145
0.6	0.0	0.02	0.01	1.5000	2.1443	6.9522
0.6	0.0	0.03	0.01	1.6667	3.0117	8.2548
0.6	0.0	0.03	0.01	1.5000	2.1334	6.9019
0.6	0.0	0.04	0.01	1.6667	2.9949	8.4396
0.6	0.0	0.04	0.01	1.5000	2.1275	6.8305
0.6	0.1	− 0.05	0.01	1.6667	3.1144	8.5751
0.6	0.1	− 0.05	0.01	1.5000	2.2215	7.1645
0.6	0.0	− 0.05	0.01	1.6667	2.9729	8.3188
0.6	0.0	− 0.05	0.01	1.5000	2.1148	6.7369

6. Conclusions and Comparison with Observations

The results are presented in Table II. All the results have been calculated by the proposed method for incorporating the effects of differential rotation and tidal distortion on the periods of small adiabatic oscillations of stellar models. The results presented in this table for undistorted case ($b_1 = b_2 = q = 0$), the tidally distorted case ($b_1 = b_2 = 0$) and distortion caused by differential rotation (i.e. $q = 0$) agree with the corresponding results available earlier in literature (cf. Mohan and Singh[23], Singh and Sharma[27]).

A comparison of these results with the corresponding results in which only rotational effects are considered shows that the eigen frequencies of differentially rotating Roche model increase (so that the corresponding periods of oscillations decrease in presence of tidal effects of companion star). However in the absence of differential rotation the results show that with the increase in the mass of the companion star causing tidal distortion these eigen values decrease (so that the periods of oscillations increase). The value of Γ also affects the eigen values, increase in the value of Γ increase the eigen value of the fundamental mode. The results show that compared to the undistorted model, the eigenfrequencies decrease with the introduction of the angular velocity of rotation both for the case of solid body rotation as well as differential rotation. The decrease is larger in the case when angular velocity is increasing from axis of rotation to outwards compared to the case in which angular velocity decrease from axis of rotation to inwards. The cases where it has partial increase and partial decrease are also presented. Our results show that none of the model is found unstable as the fundamental modes in every case are positive.

The results obtained by us in this paper can be helpful in estimating the nature of the internal structure of an observed rotating variable stars. If the observed pulsating star has the internal structure which approximates a Roche model then the eigen frequencies of oscillation of this star must reasonably agree with the theoretically computed eigen frequencies of this Roche model for the observed values of its mass, radius and the angular velocity of rotation. We have compared the observed and theoretical results in the case of three stars.

(i) 12 *Lac* : 12 Lac is a pulsating star. Saio[29] has estimated its mass 12 $M_\odot$ and radius 8.7 $R_\odot$. For this star several pulsation frequencies have been identified by Jarzebowski *et al.*[15]. The values of four identified eigen frequencies as given in Saio[28] are approximately 31.8, 32.6, 33.5 and 34.5 radians per day. If we transform the results obtained by us in this paper for differentially rotating and tidally distorted Roche model in the units of the mass and radius of this star, we get the pulsation frequencies listed in Table II(a). On comparing the four observed eigen frequencies of this star with the results of Table II(a) we find that two observed eigenfrequencies can be identified. However, in non of these models, we are able to identify all the four observed eigen frequencies. Thus on the basis of our present study, it seems unlikely that the star 12 Lac will have its internal structure based on the differentially rotating model considered in Table II(a), for specific values of the parameter of the laws of differential rotation used by us in these models.

(ii) 1 *Mon* : 1 Mon is one of δ Sct type rotating variable stars. Balona and Stabie[2] have estimated its radius as 2.2 $R_\odot$. From this fact and the general consensus of opinion of δ Sct type star at a slightly evolved stage, Ando[1] has chosen (M = 1.5 $M_\odot$, R = 2.0 $R_\odot$) and (M = 2 $M_\odot$ and R = 2.6 $R_\odot$) as the reference model of the star. He has shown that one eigen frequencies of its radial mode of oscillation is 5.34231 $\times 10^{-4}$ radian per second. Balona and Stabie[2] have earlier identified its radial mode as 5.342268 $\times 10^{-4}$ radian per second. Computed eigen frequencies by us of this star closer to the observed eigen frequencies have been underlined in Table II(b).

TABLE II(a)

Eigenfrequencies of the various radial modes of oscillations of 12 *lac* (*B*1.5 *III*, β *cep stars*), Γ = 1.5

ξ_1	b_1	b_2	Eigenfrequencies of radial modes in radian per day	
			Fundamental mode	First mode
1.0	0.0	0.00	17.0225 ($0^d.3693$)*	32.5528 ($0^d.1931$)*
1.0	0.0	0.00	16.7216 ($0^d.3759$)	32.2270 ($0^d.1950$)
1.0	0.1	0.00	16.2542 ($0^d.3867$)*	29.7730 ($0^d.2111$)*
1.0	0.1	0.10	13.3508 ($0^d.4708$)*	31.7975 ($0^d.1979$)*
1.0	0.1	0.10	11.0957 ($0^d.5664$)	27.1790 ($0^d.2313$)
1.0	0.1	0.02	13.5125 ($0^d.4652$)*	32.1042 ($0^d.1957$)*
1.0	0.04	– 0.01	13.6112 ($0^d.4647$)*	32.3489 ($0^d.1943$)*
1.0	– 0.1	0.05	13.5243 ($0^d.4647$)*	32.3588 ($0^d.1943$)*
0.5	0.1	0.10	16.5964 ($0^d.3787$)*	30.7560 ($0^d.2044$)*
0.5	0.1	0.10	14.7937 ($0^d.4249$)	28.5369 ($0^d.2203$)
0.5	0.1	0.20	16.4857 ($0^d.3812$)*	25.3335 ($0^d.2481$)*
0.5	0.1	0.20	14.4202 ($0^d.4358$)	24.5306 ($0^d.2562$)
0.5	0.1	0.02	16.4660 ($0^d.3817$)	29.8598 ($0^d.2340$)
0.5	0.1	0.02	18.6890 ($0^d.3363$)*	31.4734 ($0^d.1997$)*
0.5	0.1	– 0.02	19.7025 ($0^d.3190$)*	30.1498 ($0^d.2085$)*
0.5	0.1	– 0.02	17.7395 ($0^d.3543$)	28.7591 ($0^d.2186$)
0.5	– 0.1	0.05	17.9626 ($0^d.3499$)	28.5633 ($0^d.2201$)
0.5	0.02	0.01	18.4417 ($0^d.3408$)*	30.3299 ($0^d.2072$)*

Note :

(i) Eigenfrequencies marked with asterisk are computed for Γ = 1.6667

(ii) Eigenfrequencies which are closer to observed eigenfrequencies have been underlined.

(iii) Corresponding periods have been given in parenthesis.

(iii) 16 *Lac* : 16 Lac is considered to be a pulsating star which is the primary component of a binary system, Fitch[7] obtained its orbital elements and three pulsation frequencies. Saio[28] has estimated its mass as $12M_\odot$ and radius as 8.6 $R_\odot$ and the eigenfrequencies of three non-radial modes of pulsation are approximately 34.4, 36.8 and 37.2 radians per day. In the present study only radial modes are considered, the

TABLE IIb

Eigenfrequencies of the various radial modes of oscillations of 1 Mon star, Γ = 1.5

ξ_1	b_1	b_2	Eigenfrequencies of radial modes in radian per second	
			Fundamental mode	First mode
1.0	0.0	0.00	6.0718×10^{-4} (0^d.1198)*	1.1611×10^{-3} (0^d.0626)*
			[4.7301×10^{-4}] (0^d.1538)*	[9.0455×10^{-4}] (0^d.0804)*
1.0	0.0	0.00	5.9645×10^{-4} (0^d.1220)	1.1495×10^{-3} (0^d.0628)
			[4.6465×10^{-4}] (0^d.1566)	[8.9550×10^{-4}] (0^d.0812)
1.0	0.1	0.00	5.7977×10^{-4} (0^d.1255)*	1.0620×10^{-3} (0^d.0684)*
			[4.5166×10^{-4}] (0^d.1611)*	[8.2732×10^{-4}] (0^d.0879)*
1.0	0.1	0.1	4.7621×10^{-4} (0^d.1528)*	1.2864×10^{-3} (0^d.0565)*
			[3.7098×10^{-4}] (0^d.1961)*	[8.8357×10^{-4}] (0^d.0823)*
1.0	0.1	0.10	3.9578×10^{-4} (0^d.1838)	9.6945×10^{-4} (0^d.0750)
			[3.0832×10^{-4}] (0^d.2360)*	[7.5523×10^{-4}] (0^d.0963)
1.0	0.1	0.02	4.8120×10^{-4} (0^d.1512)*	1.1451×10^{-3} (0^d.0634)*
			[3.7580×10^{-4}] (0^d.1935)*	[8.9210×10^{-4}] (0^d.0815)*
1.0	0.04	− 0.01	4.8550×10^{-4} (0^d.1498)*	1.1539×10^{-3} (0^d.0630)*
			[3.7822×10^{-4}] (0^d.1923)*	[8.9889×10^{-4}] (0^d.0809)*
1.0	− 0.1	0.05	4.8240×10^{-4} (0^d.1508)*	1.1542×10^{-3} (0^d.0630)*
			[3.7581×10^{-4}] (0^d.1936)*	[8.9917×10^{-4}] (0^d.0809)*
.5	0.1	0.10	5.9198×10^{-4} (0^d.1228)*	1.0970×10^{-3} (0^d.0662)*
			[4.6117×10^{-4}] (0^d.1577)*	[8.6463×10^{-4}] (0^d.0851)*
.5	0.1	0.10	5.2768×10^{-4} (0^d.1379)	1.0179×10^{-3} (0^d.0714)
			[4.1180×10^{-4}] (0^d.1767)	[7.9297×10^{-4}] (0^d.0917)

(Continued on page 119)

Table IIb — Continued

ξ_1	b_1	b_2	Eigenfrequencies of radial modes in radian per second	
			Fundamental mode	First mode
.5	.1	0.20	5.8803 × 10^{-4} ($0^d.1237$)*	9.0363 × 10^{-4} ($0^d.0805$)*
			[4.5810 × 10^{-4}] ($0^d.1588$)*	[7.0395 × 10^{-4}] ($0^d.1033$)*
.5	0.1	0.20	5.1436 × 10^{-4} ($0^d.1414$)	8.7499 × 10^{-4} ($0^d.0831$)
			[4.6070 × 10^{-4}] ($0^d.1579$)	[6.8164 × 10^{-4}] ($0^d.1067$)
.5	0.1	0.02	5.8733 × 10^{-4} ($0^d.1239$)	1.0650 × 10^{-3} ($0^d.0683$)
			[4.5755 × 10^{-4}] ($0^d.1590$)	[8.2973 × 10^{-4}] ($0^d.0877$)
.5	0.1	0.02	6.6663 × 10^{-4} ($0^d.1091$)*	1.1226 × 10^{-3} ($0^d.0648$)*
			[5.1932 × 10^{-4}] ($0^d.1401$)*	[8.7456 × 10^{-4}] ($0^d.0831$)*
.5	0.1	– 0.02	7.0278 × 10^{-4} ($0^d.1035$)*	1.0754 × 10^{-3} ($0^d.0676$)*
			[5.4748 × 10^{-4}] ($0^d.1329$)*	[8.3779 × 10^{-4}] ($0^d.0868$)*
.5	0.1	– 0.02	6.3276 × 10^{-4} ($0^d.1150$)	1.0258 × 10^{-3} ($0^d.0709$)
			[4.9294 × 10^{-4}] ($0^d.1476$)	[7.9914 × 10^{-4}] ($0^d.0910$)
.5	– 0.1	0.05	6.4071 × 10^{-4} ($0^d.1135$)	1.0188 × 10^{-3} ($0^d.0714$)
			[4.9913 × 10^{-4}] ($0^d.1458$)	[7.9370 × 10^{-4}] ($0^d.0917$)
.5	0.02	0.10	6.5780 × 10^{-4} ($0^d.1106$)*	1.0818 × 10^{-3} ($0^d.0672$)*
			[5.1245 × 10^{-4}] ($0^d.1420$)*	[8.4279 × 10^{-4}] ($0^d.0863$)*

Note :

(i) Eigenfrequencies marked with asterisk are computed for $\Gamma = 1.6667$.

(ii) Results for 1 Mon star for $M = 1.5M_\odot$ and $R = 2.0R_\odot$.

(iii) Results in [] are $M = 2M_\odot$ and $R = 2.6R_\odot$ (cf. Ando).

(iv) Eigenfrequencies which are closer to observed eigenfrequencies have been underlined.

(v) Corresponding periods have been given in parenthesis.

values given in Table II(c) do not match the observed modes. This confirm the fact that perhaps in the binary system the non-radial modes are more dominant than the radial modes. This happens because of the fact that the distortion are caused in the shape of primary component due to the gravitational pull of secondary component (less massive). We intend to investigate this aspect of non-radial modes in our subsequent study.

TABLE II(c)

Eigenfrequencies of the various radial modes of oscillations of 16 lac star, $\Gamma = 1.5$

ξ_1	b_1	b_2	q	Eigenfrequencies of radial modes in radian per day	
				Fundamental mode	First mode
0.4	0.0	0.00	0.00	17.8665 ($0^d.3515$)*	29.7235 ($0^d.2113$)*
0.4	0.0	0.00	0.00	16.0404 ($0^d.3915$)	27.6087 ($0^d.2275$)
0.4	0.1	0.10	0.01	15.9136 ($0^d.3946$)*	24.1051 ($0^d.2605$)*
0.4	0.1	0.10	0.01	14.1509 ($0^d.4438$)	22.3158 ($0^d.2814$)
0.4	0.1	0.02	0.01	13.3315 ($0^d.4711$)*	22.3954 ($0^d.2804$)*
0.4	0.1	0.02	0.01	12.3297 ($0^d.5093$)	20.6981 ($0^d.3034$)
0.4	0.1	0.03	0.01	13.6860 ($0^d.4589$)*	22.0666 ($0^d.2846$)*
0.4	0.1	0.03	0.01	12.3702 ($0^d.5077$)	20.4040 ($0^d.3078$)
0.4	0.1	0.04	0.01	13.5498 ($0^d.4635$)	21.4306 ($0^d.2930$)*
0.4	0.1	0.04	0.01	12.0956 ($0^d.5192$)	19.8234 ($0^d.3168$)
0.4	0.1	− 0.05	0.01	14.5702 ($0^d.4310$)	22.1783 ($0^d.2832$)*
0.4	0.1	− 0.05	0.01	13.0627 ($0^d.4808$)	20.5816 ($0^d.3051$)
0.4	0.0	− 0.05	0.01	12.9579 ($0^d.4846$)*	20.5173 ($0^d.3061$)*
0.4	0.0	− 0.05	0.01	11.4788 ($0^d.5471$)	18.9426 ($0^d.3315$)
0.5	0.0	0.00	0.00	17.1903 ($0^d.3653$)*	27.4100 ($0^d.2291$)*
0.5	0.0	0.00	0.00	15.8065 ($0^d.3973$)	27.5677 ($0^d.2278$)
0.5	0.0	0.00	0.01	16.9206 ($0^d.3711$)*	32.6297 ($0^d.1925$)*
0.5	0.0	0.00	0.01	15.4579 ($0^d.4063$)	30.3640 ($0^d.2069$)
05	0.0	0.00	0.02	16.3364 ($0^d.3844$)*	26.7362 ($0^d.2349$)*
0.5	0.0	0.00	0.02	14.3571 ($0^d.4374$)	24.5075 ($0^d.2562$)

Note : (i) Eigenfrequencies marked with asterisk are computed for $\Gamma = 1.6667$.

(ii) Corresponding periods have been given in paranthesis.

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REFERENCES

1. H. Ando, *Ap. Space Sci.* **73** (1980), 159.
2. L. A. Balona and R. S. Stobie, 1979 (Preprint).
3. P. Bodenheimer, *Astrophys. J.* **167** (1971), 153.
4. J. P. Cox, *Theory of Stellar Pulsations*, Princeton Univ. Press, 1980.
5. K. L. Chan and W. Y. Chau, *Astrophys. J.* **233** (1979), 950.
6. M. J. Clement, *Astrophys. J.* **150** (1967), 589.
7. W. S. Fitch, *Astrophys. J.* **158** (1969), 269.
8. W. S. Fitch and W. Z. Wisniewski, *Astrophys. J.* **231** (1979), 808.
9. V. S. Geroyannis, J. N. Tokis and F. N. Valvi, *Ap. Space Sci.* **64** (1979), 359.
10. V. S. Geroyannis and G. A. Antonakopoulas, *Ap. Space Sci.* **74** (1981), 97.
11. E. Hoiland, *Avhandliger Norske Videnskaps — Akademi i*, 1941.
12. I. Oslo, *Math. naturv. Klasse*, **11**, 1.
13. Icko Iben (Jr), *Astrophys. J.* **304** (1986), 201.
14. J. G. Ireland, *Z. Astrophys.* **65** (1967), 123.
15. T. Jarzebowski *et al.*, *Acta Astron.* **29** (1979), 517.
16. Z. Kopal, *Adv. Astron. Astrophys.* **9** (1972), 1.
17. R. Kippenhahn and H. C. Thomas, in : *Stellar Rotation* (A. Slettlebak (ed.)), D. Reidel Publ. Co., Dordrecht, Holland, 1970, p. 20.
18. H. Komatsu, Y. Eriguchi and I. Hachisu, *M. N. R. A. S.* **239** (1989), 153.
19. P. Ledoux and T. Walravan, *Variable stars in the Han Buch der physik*, 51, ed., flugge, 1958, pp. 338-601.
20. L. Martin and P. Smeyers, *Astron. Astrophys.* **106** (1982), 317.
21. C. Mohan and V. P. Singh, *Astrophys. Space Sci.* **54** (1978), 293.
22. C. Mohan and V. P. Singh, *Astrophys. Space Sci.* **60** (1979), 423.
23. C. Mohan and V. P. Singh, *Astrophys. Space Sci.* **85** (1982), 83.
24. C. Mohan and R. M. Saxena; *Astrophys. Space Sci.* **95** (1983), 369.
25. C. Mohan, R. M. Saxena and S. Agarwal, *Astrophys. Space Sci.* **163** (1990), 23.
26. C. Mohan, A. K. Lal and V. P. Singh, *Astrophys. Space Sci.* **193** (1992), 69.
27. V. P. Singh and M. K. Sharma, *Indian J. pure appl. Math.* **26** (1995), 69-79.
28. H. Saio, *Ap. J.* **244** (1981), 299.
29. H. Solberg, *Process-Verbax Ass. Meteor, U. G. C. 1. 6,* Assemblee Generale (Edinburgh), Mem. et Disc, 1936
30. R. Stoeckly, *Astrophys. J.* **142** (1965), 208.
31. J. L. Tassoul and M. Tassoul, *Asrophys. J.* **261** (1982), 265.
32. J. L. Tassoul, *Theory of Rotating Stars*, Princenton University Press, Princenton, 1978.
33. I. Todaran, *Ap. Space Sci.* **187** (1992), No. 1.
34. H. Von Zeipal, *M. N. R. A. S.* **84** (1924), 665.