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TOPOLOGICALLY INVERTIBLE ELEMENTS IN METRIZABLE ALGEBRAS

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Following the definition of (boundedly) topologically invertible elements given in Bhatt and Thatte[1], we characterize the algebras in which every boundedly topologically invertible element is invertible. In particular, we show that in B_0-algebras, the boundedly topologically invertible elements are in fact invertible. This allows us to embed homeomorphically any metrizable locally convex algebra with continuous product in an algebra of the same kind in which all boundedly topologically invertible elements are invertible. This is not always possible for topologically invertible elements; so, we answer in the negative the question asked in Bhatt and Thatte[1].

Bhatt and Thatte[1] have defined the notion of (boundedly) topologically invertible element in metrizable locally convex algebras (l.c.a). They have shown that in topological Q-algebras as well as in complete locally multiplicatively convex algebras (l.m.c.a) each topologically invertible element is, in fact, invertible. So this notion is of no interest in these algebras. However, they have shown that in the Aren's algebra L^{ω}, which is a B_0-algebra, there exists topologically invertible elements that are not invertible.

In this paper, using Bhatt and Thatte[1], we characterize the metrizable l.c.a. with continuous product in which every boundedly topologically (b-topologically) invertible element is invertible (such algebras are said to satisfy property (B)). They are exactly those that are inverse closed in their completions (Proposition 3). In particular, B_0-algebras satisfy the property (B). By considering the completion, we easily obtain the main result of Bhatt and Thatte[1] which is the embedding of a metrizable l.c.a. with continuous product in an algebra of the same kind satisfying (B).

The authors have considered the algebra $[l^{\infty}(A)]$ of classes of bounded sequences in A (modulo the ideal of sequences converging to zero). The algebra A is homeomorphically embedded in $[l^{\infty}(A)]$. They have asserted that $[l^{\infty}(A)]$ satisfies the property (B), Bhatt and Thatte[1] (Proposition, p. 1311). We show that their proof is false. However, the result remains true; in fact we show that $[l^{\infty}(A)]$ is a B_0-algebra whenever A is a (not necessarily complete) metrizable l.c.a. with continuous product. They have also suggested to modify their embedding procedure that may make all topologically invertible elements invertible. We show that this is not always possible. In fact we show that algebras admitting topologically non b-topologically invertible elements can't be embedded in an algebra satisfying property (T) (see Definition 1).

Throughout the sequel, the considered algebras are complex, associative and unital. The unit is noted e. By a topological algebra A, we mean an algebra which is a topological space in such a way that the product is separately continuous. If the product is jointly continuous, we speak of a topological algebra with continuous product. If the linear space A is locally convex, we say that A is a locally convex algebra (l.c.a) (its topology can be defined by a family $(p_\lambda)_{\lambda \in \Lambda}$ of semi norms). If $(p_\lambda)_{\lambda \in \Lambda}$ satisfies : $p_\lambda(xy) \leq p_\lambda(x)\, p_\lambda(y)$, for every $x, y \in A$ and every $\lambda \in \Lambda$, we say that A is locally multiplicatively convex algebra (l.m.c.a). A B_0-algebra is a complete metrizable l.c.a. A topological algebra A is called a Q-algebra if the set of invertible elements $G(A)$ is open. It is said to be inverse closed in its completion $\tilde{A}$ if $G(\tilde{A}) \cap A = G(A)$. By an A-normed algebra, we mean a topological algebra whose topology is defined by one linear norm.

Throughout the sequel, A will denote a metrizable l.c.a. with continuous product and $(p_n)_n$ an increasing family of semi-norms defining the topology of A.

Definition 1 — An element $x \in A$ is called topologically invertible if there exists a sequence $(x_n)_n$ in A such that $(xx_n)_n$ and $(x_nx)_n$ both converge to the unit element e. If furthermore $(x_n)_n$ is bounded (resp, a Cauchy sequence), x is called b-topologically (resp, C-topologically) invertible. We say that A satisfies the property (T) (resp, the property (B)) if every topologically (resp, b-topologically) invertible element of A is invertible.

Examples — (1) Every topological Q-algebra and every complete l.m.c.a. satisfies (T) (Bhatt and Thatte[1]).

(2) The Arens's algebra L^ω contains topologically invertible non-invertible elements[1]. It doesn't satisfy (T).

(3) Let A be the algebra of complex polynomial functions endowed with the topology of uniform convergence on [0, 1]. Let P be a non-constant element of A having no zero in [0, 1]. By the Stone-Weierstrass theorem, there exists a sequence $(P_n)_n$ in A converging to the inverse of P in the algebra of continuous functions on [0, 1]. So $(PP_n)_n$ converges to 1 in A and $(P_n)_n$ is bounded. Thus P is b-topologically invertible non-invertible in A.

Proposition 2 — If A is a l.m.c. algebra satisfying property (B), then it satisfies property (T).

PROOF : Let $x \in A$ and $(x_n)_n$ be a sequence in A such that $(xx_n)_n$ and $(x_nx)_n$ converge to e in A. Therefore they converge to e in the completion $\tilde{A}$ of A. Since A satisfies (T), one can deduce that $(x_n)_n$ is convergent in $\tilde{A}$. So it is bounded in A.

Bhatt and Thatte[1] have shown that : $x \in A$ is b-topologically invertible in A if and only if x is invertible in $[l^\infty(A)]$. Using this result, we show that one has in fact :

Proposition 3 — (1) Let $x \in A$. Then x is b-topologically invertible in A if and only if x is invertible in the completion $\tilde{A}$ of A.

(2) A satisfies property (B) if and only if A is inverse closed in its completion $\tilde{A}$

PROOF : Let $(x_n)_n$ be a bounded sequence in A such that $(xx_n)_n$ and $(x_nx)_n$ converge to e in A, they also converge to e in $[l^\infty(A)]$. By Bhatt and Thatte[1], x is invertible in $[l^\infty(A)]$. Therefore $(x_n)_n$ converges to x^{-1} in $[l^\infty(A)]$. We deduce from this that $(x_n)_n$ is a Cauchy sequence in A and $x^{-1} \in \tilde{A}$. Conversely, if x is invertible in $\tilde{A}$, then there exists a Cauchy sequence $(x_n)_n$ in A such that $(xx_n)_n$ and $(x_nx)_n$ converge to e in A. The assertion (2) follows from (1).

Remark 4 : As a consequence of the previous proposition, one sees that normed algebras satisfying (B) are Q-algebras. This last result fails to be true for the class of A-normed algebras. Indeed, consider the algebra of complex measurable functions f on [0, 1] of the form $f = \sum_{i=1}^{n} \lambda_i \chi_{A_i}$; $n \in \mathbb{N}$ and $(A_i)_{1 \le i \le n}$ is a partition of [0, 1] such that meas $(A_i) > 0$, for every $i = 1, 2, ..., n$. This algebra endowed with the usual function operations and with the linear norm $\|f\|_1 = \int_{[0,1]} |f(t)|\, dt$ is an A-normed algebra. It satisfies (T) but this is not a Q-algebra.

Proposition 5 — If A can be homeomorphically embedded in an algebra B in which all topologically invertible elements in A are invertible (in particular, if A is l.m.c), then, for all $x \in A$, the following assertions are equivalent : (1) x is topologically invertible in A. (2) x is b-topologically invertible in A. (3) x is C-topologically invertible in A. (4) x is invertible in the completion $\tilde{A}$ of A.

PROOF : Clearly it suffices to show : (1) $\Rightarrow$ (4). One proceeds in the same manner as in Proposition 3 replacing $[l^\infty(A)]$ by B.

In the abstract of Bhatt and Thatte[1], it is asserted that A can be homeomorphically embedded in another algebra in which all topologically invertible elements are invertible. The previous proposition shows that this assertion is incorrect. It solves, furthermore, the question asked in Bhatt and Thatte[1], section 9.

It is also asserted (see Bhatt and Thatte[1], Proposition in section 8) that $[l^\infty(A)]$ satisfies (B). The proof of this affirmation is false. Indeed, the authors take an element $[(x_n)_n]$ of $[l^\infty(A)]$ and $([(y_n^{(k)})_n])_k$ a bounded sequence in $[l^\infty(A)]$ such that $[(x_n)_n] \cdot [(y_n^{(k)})_n]$ converge to $[e]$ in $[l^\infty(A)]$. Then, they consider a continuous semi-norm p (fixed). Since $\limsup_n p(x_n y_n^{(k)} - e)$ converges to 0, they assert that :

(*) For every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $p(x_n \cdot y_n^{(k_n)} - e) < \frac{1}{n}$.

(*) is false. Indeed, choose $A = \mathbb{C}$, the field of complex numbers and $x_n = 1 + \frac{2}{n}$, $y_n^{(k)} = 1$, for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$. One has : $[(x_n)_n] \cdot [(y_n^{(k)})_n] = [1]$, for all k. But for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$, $\left|\left(1+\frac{2}{n}\right)\cdot 1-1\right| = \frac{2}{n} > \frac{1}{n}$. Another remark, to mention, is that, they put $z_n = y_n^{(k_n)}$ and from (*) they conclude that $(x_n z_n)_n$ converge to e, forgetting that the sequence $(z_n)_n$ is related to the fixed semi-norm p. However, their assertion remains true. In fact one has :

Proposition 6 — For all metrizable l.c.a. A with continuous product, the algebra $[l^\infty(A)]$ is a B_0-algebra. In particular $[l^\infty(A)]$ satisfies property (B).

PROOF : Let $\tilde{A}$ be the completion of A. The topology of $\tilde{A}$ is defined by $(\tilde{p}_m)_m$ (Horvath[3]), where $\tilde{p}_m$ is the extension of p_m to $\tilde{A}$. Consider $l^\infty(\tilde{A})$, the algebra of bounded sequences in $\tilde{A}$. Endowed with the family of semi-norms $\|(x_n)_n\|_m = \sup\{\tilde{p}_m(x_n),\ n \geq 0\}$, it is a B_0-algebra. The ideal $\tilde{C}_0$ of sequences converging to x in $\tilde{A}$ being closed, $[l^\infty(\tilde{A})]$ with the quotient topology is also a B_0-algebra. Consider the mapping :

$$\Phi : [l^\infty(A)] \to [l^\infty(\tilde{A})]$$

$$(x_n)_n + C_0 \mapsto (x_n)_n + \tilde{C}_0$$

It is clear that Φ is a one to one homomorphism. We shall show that Φ is surjective. Let $(x_n)_n + \tilde{C}_0 \in [l^\infty(\tilde{A})]$. Since A is dense in $\tilde{A}$, there exists, for every $n \in \mathbb{N}$, $z_n \in A$ such that : $\tilde{p}_n(x_n - z_n) \leq \frac{1}{n}$. Let $m \in \mathbb{N}$. For $n \geq m$, one has : $p_m(z_n) = \tilde{p}_m(z_n) \leq \tilde{p}_m(z_n - x_n) + \tilde{p}_m(x_n)$ and since $(\tilde{p}_m)_m$ is increasing, we have : $\tilde{p}_m(z_n) \leq \tilde{p}_n(z_n - x_n) + \tilde{p}_m(x_n)$. Thus : $p_m(z_n) \leq \frac{1}{n} + \tilde{p}_m(x_n)$. The sequence $(z_n)_n$ is therefore bounded in A; and since $\tilde{p}_m(z_n - x_n) \leq \frac{1}{n}$, one has : $(z_n)_n + \tilde{C}_0 = (x_n)_n + \tilde{C}_0$. i.e. $\Phi((z_n)_n + C_0) = (x_n)_n + \tilde{C}_0$. Thus $[l^\infty(A)]$ is algebraically isomorphic to $[l^\infty(\tilde{A})]$. Since the topology of $[l^\infty(A)]$ (defined in Bhatt and Thatte[1]) coincides with the quotient topology of $l^\infty(A)$ by C_0, $[l^\infty(A)]$ and $[l^\infty(\tilde{A})]$ are isometrically isomorphic. So $[l^\infty(A)]$ is a B_0-algebra. This completes the proof.

Example — Consider the algebra A of Remark 4. We endow A with the topology defined by the family of semi-norms $(\|\ \|_n)_n$; where $\|f\|_n = [\int_{[0,1]} |f|^n\, dt]^{1/n}$. $(A, (\|\ \|_n)_n)$ is not complete (its completion is the Arens's algebra L^ω). It satisfies (B). Indeed, it suffices to show that it is inverse closed in L^ω (Proposition 3). Let $g \in L^\omega$ and $f = \sum_{i=1}^{i=n} \lambda_i \chi_{A_i} \in A$ such that $f.g = 1$.

So $f(x) \cdot g(x) = 1$, for almost every $x \in [0, 1]$. Thus $\lambda_i \neq 0$, $\forall i$. Therefore $g = \sum_{i=1}^{i=n} (1/\lambda_i)\chi_{A_i}$ is an element of A.

Final remarks : (1) One can extend the definition of topologically invertible elements to general topological algebras by replacing sequences by generalized sequences. One can show that Q-topological algebras and advertibly complete l.m.c. algebras[4, 5] satisfy (T).

(2) The algebra $C_b(\mathbb{R})$ of complex bounded continuous functions on $\mathbb{R}$, endowed, with the strict topology β (Buck[2]) is a non metrizable complete l.c. algebra. It is not multiplicatively convex (Buck[2]). It contains topologically invertible elements non-invertible in $C_b(\mathbb{R})$. Indeed, consider, for example, the element of $C_b(\mathbb{R})$ defined by :

$$f(x) = \frac{1}{1+x^2} \quad \text{if } x \geq 0 \text{ and } f(x) = 1 \text{ if } x \leq 0.$$

Then $(ff_n)_n$ converges to 1 in $(C_b(\mathbb{R}), \beta)$; where $f_n(x) = 1 + x^2$ if $0 \leq x \leq n$, $f_n(x) = 0$ if $n + 1 \leq x$, $f_n(x) = 1$ if $x \leq 0$ and f_n is linear on $[n, n+1]$.

REFERENCES

1. S. J. Bhatt and A. D. Thatte, *Indian J. pure appl. Math.* **15** (12) (1984), 1308-12.
2. B. C. Buck, *Michigan Math. J.* **5** (1958), pp. 96-104.
3. J. Horvath, *Topological Vector Spaces and Distributions*, Vol. I, Addison-Wesley Publ. Co., Reading, Massachusetts, 1966.
4. A. Mallios, *Topological Algebras Selected topics*, North-Holland, Amsterdam, 1986.
5. S. Warner, *Duke Math. J.* **23** (1956), 1-11.