Indian J. pure appl. Math., 27(2) : 129-140, February 1996

MODULES, ANNIHILATORS AND MODULE DERIVATIONS OF JB^*-ALGEBRAS

S. HEJAZIAN AND A. NIKNAM

Department of Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159-91775, *Mashhad, Iran*

(*Received* 11 *July* 1995; *accepted* 6 *October* 1995)

In this paper we study modules and module derivations of JB^*-algebras. We prove some results for annihilators of submodules, and we also prove the continuity of module derivations of JB^*-algebras to Banach Jordan modules in certain cases which is a generalization of results in Ringrose[9]. Our method works in particular for commutative C^*-algebras and the C^*-algebra $K(H)$, of compact operators on a Hilbert space H. We also show the existence of discontinuous module derivations of certain JB^*-algebras.

INTRODUCTION

Let A be a Jordan algebra and X be a vector space over the same field as A. X is said to be a Jordan A-module, if there is a pair of bilinear mappings $(a, x) \to a \circ x$ and $(a, x) \to x \circ a$ of $A \times X$ to X, such that, for each $a, b \in A$ and $x \in X$, the following hold :

$$a \circ x = x \circ a$$

$$(x \circ a^2) \circ a = (x \circ a) \circ a^2$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a)(a \circ b) + (x \circ b) \circ a^2.$$

Let $A \oplus X$, be the vector space direct sum of A and X. Then $A \oplus X$ with product $(a_1 + x_1) \circ (a_2 + x_2) = a_1 \circ a_2 + a_1 \circ x_2 + a_2 \circ x_1$, would be a Jordan algebra. It is called the split null extension of A and X, by the above bilinear mapping[6]. A linear subspace S of X is called a submodule, if $\{a \circ x \mid a \in A \text{ and } x \in S\} \subseteq S$. It is easy to see that, if A is an associative algebra, and X is a bi-module of A, with module operations, $(a, x) \to ax$ and $(a, x) \to xa$, then the bilinear mapping $(a, x) \to a \circ x = \frac{1}{2}(ax + xa)$, makes X a Jordan A^+-module, where A^+ is A equipped with the product $a \circ b = \frac{ab + ba}{2}$. Let A be a Banach Jordan algebra, and X a Banach space which is also a Jordan A-module, then X is called a Jordan A-module with continuous module operation, if the mapping $x \to x \circ a = a \circ x$, from X to X, is continuous for all $a \in A$. X is said to be a Banach Jordan A-module, if there exists $M \geq 0$, such that $\| a \circ x \| \leq M \| a \| \| x \|$ $(a \in A,\ x \in X)$. For example A^*, the topological dual of A, is a Banach Jordan A-module, with module operation $(a, f) \to a \circ f$, defined by $(a \circ f)(b) = f(a \circ b) \quad (a, b \in A)$.

1. Annihilators for Submodules

Let A be a Jordan algebra and X be a Jordan A-module. For each $a \in A$, we denote by R_a and U_a, the linear operators $u \to a \circ u$ and $u \to 2a \circ (a \circ u) - a^2 \circ u$ on $A \oplus X$, respectively. Given a submodule S of X, take $R(S) = \{a \in A \mid R_a(S) = \{0\}\}$, then $R(S)$ is a linear subspace of A. Let $J(S)$ be the largest ideal of A contained in $R(S)$, then it is easy to see that $J(S) = \{a \in R(S) \mid a \circ b \in R(S),\ \text{for all } b \in A\}$ (Zelmanov[14]). As in Zelmanov[14], $J(S)$ is called the annihilator of S in A. We define another annihilating set for S to be the set $\tau(S) = \{a \in A \mid U_a(S) = \{0\}\}$, which is called the quadratic annihilator of S. Since for all elements a and b in a Jordan algebra, $U_{a^2} = U_a^2$ and $U_{U_a b} = U_a U_b U_a$, then for each $a \in \tau(S)$, we have $a^2 \in \tau(S)$, $U_{\tau(S)} A \subset \tau(S)$ and $U_A \tau(S) \subseteq \tau(S)$. We also have $\tau(S) \cap R(S) = \{a \in R(S) \mid a^2 \in R(S)\}$. Moreover if B is a subalgebra of A contained in $R(S)$, then $B \subseteq \tau(S)$. We recall that for all elements a, b and c in a Jordan algebra the Jordan triple product is defined by :

$$\{abc\} = (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b$$

and if d is an element in this algebra then we have :

$$\{abc\} \circ d = \{(a \circ d)\, bc\} + \{ab\, (c \circ d)\} - \{a(b \circ d)c\}$$

(see Hanche-Olsen[5], 2-42).

Lemma 1.1 — Let A be a Jordan algebra and X be a Jordan A-module. If S is a submodule of X then :

(i) For $a, b \in R(S)$ and $c \in A$ we have $\{abc\} \circ x = (a \circ b) \circ (c \circ x)$ $(x \in S)$

(ii) If $a, b, c \in R(S)$ then $\{abc\}$, $U_a b$ and a^{2k+1} $(k = 0, 1, 2, \ldots)$ also belong to $R(S)$.

(iii) If $a \in R(S) \cap \tau(S)$ then $a^n \in R(S) \cap \tau(S)$ for $n = 1, 2, \ldots$.

Proof : (i) By considering the split null extension $A \oplus X$, if $a, b \in R(S)$ and $c \in A$ then for each $x \in S$ we have

$$\{abc\} \circ x = \{ab(c \circ x)\} = (a \circ b) \circ (c \circ x)$$

since S is a submodule, and $c \circ x \in S$.

(ii) If in (i) $c \in R(S)$ then $c \circ x = 0$ hence

$$\{abc\} \circ x = (a \circ b) \circ (c \circ x) = 0.$$

Therefore $\{abc\} \in R(S)$. Take $a = c$, then $U_a b = \{aba\} \in R(S)$, and if $a = b = c$, then $a^3 \in R(S)$ for all $a \in R(S)$. But $a^{2k+1} = U_a a^{2k-1}$ and by induction we can see that $a^{2k+1} \in R(S)$, for $k = 0, 1, 2, \ldots$

(iii) If $a \in \tau(S) \bigcap R(S)$ then $a^2 \in \tau(S) \bigcap R(S)$. Since $\tau(S) \bigcap R(S) = \{a \in R(S) \mid a^2 \in R(S)\}$. From the inclusion $U_a(A) \subseteq \tau(S)$, we have $a^n \in \tau(S)$ for $n = 1, 2, \ldots$. By (i) and the fact that $a^2 \in \tau(S) \bigcap R(S)$, again by induction we can show that $a^n \in \tau(S) \bigcap R(S)$, for $n = 1, 2, \ldots$. ■

It is easy to see that if A is a Banach Jordan algebra and X is a Banach Jordan A-module, then for each submodule $S \subseteq X$, we have $R(S)$, $\tau(S)$, and $J(S)$ are closed. In this section we are going to show that if A is a JB-algebra[5], then $J(S) = \tau(S) \bigcap R(S)$, and this will extend some of the results in Bataglia[2], for submodules. We will also use these results to prove continuity of some linear operators of JB^*-algebras in the next section.

Lemma 1.2 — Let A be a JB-algebra and X be a Banach Jordan A-module. Suppose that $S \subseteq X$ is a submodule of X. If $a \in \tau(S) \bigcap R(S)$ then $C(a) \subseteq \tau(S) \bigcap R(S)$. ($C(a)$ is the closed subalgebra of A generated by a.)

PROOF : By Lemma 1.1 (iii), if $a \in \tau(S) \bigcap R(S)$, $a^n \in \tau(S) \bigcap R(S)$ for $n = 1, 2, \ldots$. Hence any polynomial of a, without constant term lies in $R(S)$, since $R(S)$ is a closed linear subspace of X, and it follows that $C(a) \subseteq R$. On the other hand $C(a) \subseteq \tau(S)$. Therefore $C(a) \subseteq \tau(S) \bigcap R(S)$. ■

Lemma 1.3 — Suppose that A is a JB-algebra and X is a Banach Jordan A-module. If $S \subseteq X$ is a submodule of X, then for each $a \in R(S) \bigcap \tau(S)$ and $c \in R(S)$ we have $a \circ c \in R(S)$.

PROOF : By Lemma 1.1 in (i), $\{abc\} \circ x = 0$ $(a, b, c \in R(S))$. Take $b = c$ then

$$b^2 \circ c \in R(S) \qquad (b, c \in R(S)). \qquad \ldots (1)$$

If $b \in \tau(S) \bigcap R(S)$ and $b \geq 0$ then by Lemma 1.2 and (1) we have $b \circ c \in R(S)$. Now if $a \in \tau(S) \bigcap R(S)$ then $a = a_1 - a_2$ where $a_1, a_2 \geq 0$ and $a_1, a_2 \in C(a)$. The above argument shows that $a \circ c \in R(S)$, for all $a \in \tau(S) \bigcap R(S)$ and $c \in R(S)$. ■

Theorem 1.4 — Let A be a JB-algebra, and X be a Banach Jordan A-module. If $S \subseteq X$ is a submodule of X then $J(S) = \tau(S) \bigcap R(S)$.

PROOF : First we show that $\tau(S) \cap R(S)$ is a linear subspace of A. Let $a, b \in \tau(S) \cap R(S)$ and $\alpha, \beta \in \mathbb{R}$. Obviously $\alpha a + \beta b \in R(S)$. For each $x \in S$ we have:

$$U_{\alpha a + \beta b}(x) = 2\alpha\beta\, [R_a R_b (x) + R_b R_a (x) - R_{a \circ b}(x)]$$

$$+ \alpha^2 U_a (x) + \beta^2 U_b (x) = -2\alpha\beta\, (a \circ b) \circ x.$$

By Lemma 1.3, $a \circ b \in R(S)$, hence $U_{\alpha a + \beta b}(x) = 0$, that is, $\alpha a + \beta b \in \tau(S) \cap R(S)$. Now let $c \in A$, by Lemma 1.1 (i), and Lemma 1.3 for a, $b \in \tau(S) \cap R(S)$ and $x \in S$, $\{abc\} \circ x = 0$, hence $\{abc\} \in R(S)$. If $a = b$ then $a^2 \circ c \in R(S)$ for each $a \in \tau(S) \cap R(S)$ and $c \in A$, and the same argument as in Lemma 1.3, shows that for each $a \in \tau(S) \cap R(S)$, we have $a \circ c \in R$. Hence

$$\tau(S) \cap R(S) \subseteq \{a \in R(S) \mid a \circ c \in R(S) \text{ for all } c \in A\} = J(S)$$

But we have $J(S) \subseteq \tau(S) \cap R(S)$. Thus we have the result. ■

Corollary 1.5 — Using the same notation as in Bataglia[2] for each closed ideal I, in a JB-algebra A, $I^{\perp}$ is a closed ideal, and $I^{\perp} = \tau(I)$.

PROOF : Using the same identities as in Lemma 3.2 of Bataglia[2]. We have $\tau(I) \subseteq R(I)$. ■

Theorem 1.6 — Let A be a JB-algebra, and A^* be the topological dual of A. If we consider A^*, as a Banach Jordan A-module, then for each submodule $S \subseteq A^*$, $\tau(S)$ is an ideal of A.

PROOF : Given $a \in \tau(S)$, for each $f \in S$, we have $U_a f = 0$. Therefore $2a \circ (a \circ f) = a^2 \circ f$. Suppose that $\{e_\alpha\}$ is a bounded approximate identity for A, Alfsen *et al.*[1] then $2(a \circ (a \circ f))(e_\alpha) = (a^2 \circ f)(e_\alpha)$ for each α. Thus $\lim_\alpha 2f(a \circ (a \circ e_\alpha)) = \lim_\alpha f(a^2 \circ e_\alpha)$, and it follows that $2f(a^2) = f(a^2)$. Hence $f(a^2) = 0$ for all $f \in S$ and $a \in \tau(S)$. Let $b \in A$ then we have :

$$(a^2 \circ f)(b) = f(a^2 \circ b) = b \circ f(a^2) = 0 \qquad (a \in \tau(S),\ f \in S).$$

Thus $a^2 \circ f = 0$ and $a^2 \in R(S)$. It follows that any positive element of $\tau(S)$ lies in $R(S)$. Thus $\tau(S) \subseteq R(S)$. By Theorem 1.4, $\tau(S) = J(S)$. ■

In the next stage we will show that in a JB-algebra, the quadratic annihilator of a closed submodule S, of a Banach Jordan A-module is hereditary, that is, if $a \in \tau(S)$, $a \geq 0$ and $0 \leq b \leq a$ then $b \in \tau(S)$. Recall that if a JB-algebra is not unital, we can adjoin 1 to it and obtain a unital JB-algebra (Hanche-Olsen and Stormer[5], Chap. 3). We recall that a JB^*-algebra B, is a complex Banach Jordan algebra, equipped with an algebra involution *, such that $\| a . \| = \| a^* \|$, and $\| U_a a^* \| = \| a \|^3$ for each element a in this algebra. If $a = a^*$, then the closed subalgebra of B generated by a denoted by $C^*(a)$, is a C^*-algebra. The next lemma is a Jordan version of Lemma 1.4.5 of Pedersen[8].

Lemma 1.7 — Let A be a unital JB-algebra, and $a, b \in A$, be such that, $b \geq 0$ and $a^2 \leq b$, then for each real number $0 \leq \alpha \leq \frac{1}{8}$ there exist $u_\alpha \in A$, such that $a = U_{b\alpha} u_\alpha$ and $\| u_\alpha \| \leq 2 \| b^{(1/2)-2\alpha} \|$. If K is a closed ideal of A and $a, b \in K$, then $u_\alpha \in K$.

PROOF : Consider the JB^*-algebra $\mathbf{A} = A + iA$ (Wright[13]). Since a and b are self-adjoint elements in $\mathbf{A}$, then the closure of the complex Jordan subalgebra of $\mathbf{A}$ generated by a, b and 1 is a JC^*-algebra Alfsen *et al.*[1]. Let B be this JC^*-algebra. We prove the lemma for B. There exists a C^*-algebra C, such that B is a closed linear *-subspace of C, which is closed under the Jordan product $d \circ b = \frac{db + bd}{2}$ for $b, d \in B$. We have $a = a^+ - a^-$, where a^+, $a^- \geq 0$ and $a^+ a^- = a^- a^+ = 0$ in C. Moreover a^+ and a^- belong to $C^*(a)$. Since $a^2 = (a^+)^2 - (a^-)^2 \leq b$ it follows that $(a^+)^2 \leq b$ and $(a^-)^2 \leq b$. Thus $a^+ \leq b^{1/2}$ and $a^- \leq b^{1/2}$. For each $n \in \mathbf{N}$ take $c_n = \left(b + \frac{1}{n} \right)^{-\alpha} (a^+)^{1/2}$, Thus $c_n \in C$ and :

$$c_n c_n^* = \left(b + \frac{1}{n} \right)^{-\alpha} a^+ \left(b + \frac{1}{n} \right)^{-\alpha} = U_{(b+(1/n))^{-\alpha}} (a^+) \qquad \text{... (1)}$$

Therefore $$\| c_n \|^2 \leq \left\| \left(b + \frac{1}{n} \right)^{(1/2)-2\alpha} \right\| \qquad \text{... (2)}$$

By the same argument as in (3.1) of Ringrose[9] we can show that $\{c_n\}$ is a Cauchy sequence. Hence there exists $c \in C$ such that $c_n \to c$. By (2), $\| c \|^2 \leq \| b^{(1/2)-2\alpha} \|$. By (1) we have $c_n c_n^* \in B$, hence $cc^* \in B$. We also have $(a^+)^{1/2} = \left(b + \frac{1}{n} \right)^{\alpha} c_n$, let $n \to \infty$, then $(a^+)^{1/2} = b^\alpha c$. Therefore $a^+ = b^\alpha cc^* b^\alpha = U_{b\alpha} cc^*$. Similarly we may find $d \in C$, such that, $\| d \| \leq \| b^{(1/2)-2\alpha} \|$, $a^- = U_{b\alpha} dd^*$ and $dd^* \in B$. Hence $a = a^+ - a^- = U_{b\alpha} (cc^* - dd^*)$. Take $u_\alpha = cc^* - dd^*$, then $\| u_\alpha \| \leq 2 \| b^{(1/2)-2\alpha} \|$ and $a = U_{b\alpha} u_\alpha$. If K is a closed ideal and $a, b \in K$, then the above argument shows that cc^*, $dd^* \in K$. Hence $u_\alpha \in K$. ∎

Theorem 1.8 — If A is a JB-algebra and X is a Banach Jordan A-module, then for each submodule $S \subseteq X$, $\tau(S)$ is hereditary. Moreover if $a, b \in \tau(S)$ and $0 \leq a \leq b$ then $\alpha a + (1-\alpha) b \in \tau(S)$ for each $0 \leq \alpha \leq 1$.

PROOF : Let $b \in \tau(S)$ and $b \geq 0$. If $0 \leq a \leq b$, then by the previous lemma, there exists $u \in A$, such that, $a^{1/2} = U_{b^{1/16}} u$. Since $C(b) \subseteq \tau(S)$, then $b^{1/16} \in \tau(S)$. Therefore $a^{1/2}$ and hence a, lies in τ. Since $0 \leq \alpha a + (1-\alpha)b \leq 2b$, we deduce that $\alpha a + (1-\alpha) b \in \tau(S)$.

2. Module Derivations of JB^*-Algebras

Let A be a Jordan algebra and X a Jordan A-module, a linear mapping $D : A \to X$ is called a module derivation, or simply a derivation, if

$$D(a \circ b) = a \circ D(b) + Da \circ b \ (a,\ b \in A).$$

Continuity of derivations of Banach algebras, is an interesting part of automatic continuity. Recently Villena[12] extended the results of Johnson and Sinclair[7] to semi-simple Banach Jordan algebras. Ringrose[9] proved that, any module derivation of a C^*-algebra to a Banach module is continuous. In this section we study derivations of JB^*-algebras to Banach Jordan modules, and prove the continuity of these derivations in certain cases. Let D be a derivation of a JB^*-algebra A, to a Banach Jordan A-module X. Take $\sigma = \{x \in X \mid \exists\ \{a_n\} \subseteq A,\ \ a_n \to 0 \text{ and } Da_n \to x\}$, σ is called the separating space of D, and it is easy to see that σ is a closed submodule of X. Let $R(\sigma)$, $\tau(\sigma)$ and $J(\sigma)$ be defined as in section 1. Since $A = A_h + iA_h$, where $A_h = \{a \in A \mid a = a^*\}$ is a JB-algebra, then the results of Lemma 1.2 show that :

$$J(\sigma) = (\tau(\sigma) \cap R(\sigma))_h + i(\tau(\sigma) \cap R(\sigma))_h.$$

Standard theorems in automatic continuity[11] imply that :

$$R(\sigma) = \{a \in A \mid R_a D \text{ is continuous}\} = \{a \in A \mid DR_a \text{ is continuous}\}$$

and

$$\tau(\sigma) = \{a \in A \mid U_a D \text{ is continuous}\} = \{a \in A \mid DU_a \text{ is continuous}\}.$$

For more convenience we will use R, τ, and J instead of $R(\sigma), \tau(\sigma)$ and $J(\sigma)$, respectively. For a Jordan algebra B, a linear mapping T from a Jordan B-module X, to a Jordan B-module Y is called a module homomorphism, if $T(a \circ x) = a \circ T(x)$ for all $a \in B$ and $x \in X$. Return to the previous discussion, the derivation D, determines a homomorphism $\theta : A \to A \oplus X$, defined by $\theta(a) = a + Da$. Define a norm $\|\ \|_0$ on $A \oplus X$, as $\| a + x \|_0 = \| a \| + \| x \|$. This is a complete norm on $A \oplus X$, but not necessarily an algebra norm. In fact for each $u_1, u_2 \in A \oplus X$, $\| u_1 \circ u_2 \| \leq M \| u_1 \| \cdot \| u_2 \|$, where M is the bound of the bilinear mapping $(a,\ x) \to a \circ x$ of $A \times X$ to X. An easy argument shows that D is continuous iff θ is continuous. We can consider $\theta(A)$, as a Jordan A-module, with module operation $a \circ \theta(b) = \theta(a) \circ \theta(b)$ $(a, b \in A)$, and we naturally extend this operation to the closure of $\theta(A)$ in $\|\ \|_0$, say $\overline{\theta(A)}$. Then $\overline{\theta(A)}$ is a Jordan A-module with continuous module operation. Let $a \in A_h$, and $\{f_n\} \subseteq C^*(a)$ be such that, $f_n f_m = 0$ $(n \neq m)$. Then by considering the strong associativity of $C^*(a)$ in $A \oplus X$, we have $U_{f_n} U_{f_m} = U_{f_n} U_{f_m} = 0$ $(n \neq m)$, on $A \oplus X$. The same argument as in Lemma 9.1 of Sinclair[11] shows that $U_{f_n^2}\theta$, and hence $U_{f_n^2}D$ is continuous for all but a finite number of n, that is, $f_n^2 \in \tau$ for all but a finite number of n.

Theorem 2.1 — Let A be a JB^*-algebra, and X a Banach Jordan A-module, if $D : A \to X$ is a derivation, then the following conditions hold :

(i) Let $a \in A_h$, and Ω denote the spectrum of $C^*(a)$. Then the set $F = \{\lambda \in \Omega \mid \lambda(\tau \cap C^*(a)) = 0\}$ is finite.

(ii) If I is a closed ideal of A containing τ, then every element in $(A/I)_h$ has finite spectrum.

(iii) If K is an ideal of A, contained in τ, then $D|_K$ is continuous.

(iv) If L is an ideal of A such that $D|_L$ is continuous then $L \subseteq J$.

PROOF : (i) Suppose that F is infinite, and $\{\lambda_k\} \subseteq F$ is an infinite sequence, then we can find a sequence $\{V_k\}$ of open subsets of Ω such that $V_k \cap V_j = \phi$ $(k \neq j)$ and $\lambda_k \in V_k$. Choose $f_k \in C_1^*(a)$, such that $f_k(\lambda_k) \neq 0$ and $f_k \Omega \setminus V_k) = \{0\}$, for k = 1, 2, ...

Then $f_k f_j = 0$ when $k \neq j$ and $f_k^2 \notin \tau(\sigma)$, and this is a contradiction.

(ii) Let I be an ideal of A, such that, $\tau \subseteq I$. For each $a \in A$

$$\{\lambda \in \Omega \mid \lambda(I \cap C_1^*(a)) = \{0\}\} \subseteq \{\lambda \in \Omega \mid \lambda(\tau \cap C_1^*(a)) = \{0\}\}.$$

Hence $C_1^*(a)/C_1^*(a) \cap I$ is finite dimensional, and since the closed $*$-subalgebra of A/I generated by a and 1, is isomorphic to $C^*(a)/C^*(a) \cap I$, then we have the result.

(iii) It suffices to show that D is bounded on bounded subsets of K_h. Suppose on the contrary that there exists a sequence $\{a_n\} \subseteq K_h$, such that, $a_n \to 0$ and $\|D a_n\| \to \infty$. We can assume that $\sum_{n=1}^{\infty} \|a_n\|^2 \leq 1$.

Set $b = \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/8}$, then $b \geq 0$ and $a_n^2 \leq b^8$ for all $n \in N$. Moreover $\|b\| \leq 1$. By Lemma 1.7, for each $n \in N$ there exists $u_n \in K_h$, such that, $\|u_n\| \leq 2\|b^{1/4}\| \leq 2$ and $a_n = U_b(u_n)$. Thus $D(a_n) = DU_b(u_n)$. Since $K \subseteq \tau$, we have $b \in \tau$ and DU_b is bounded. Hence $\{\|Da_n\|\}$ is bounded and this is a contradiction.

(iv) Let L be an ideal in A, such that $D|_L$ is continuous. Take $y \in \sigma$. There exists $\{a_n\} \subseteq A$, such that, $a_n \to 0$ and $Da_n \to y$. For $a \in L$, we have $D(a \circ a_n) \to a \circ y$. But $a \circ a_n \in L$ and $D|_L$ is continuous, hence $a \circ y = 0$, that is, $a \in R$. Since a is arbitrary, we have $L \subseteq R$, thus $L \subseteq J$. ■

Part (iv) of the above theorem shows that J is the largest ideal of A, such that D is continuous on it. We call J the continuity ideal of D. From Theorem 1.4, $J = (R \cap \tau)_h + i(R \cap \tau)_h$.

Theorem 2.2 — Let D be a derivation of a JB^*-algebra A, to a Banach Jordan A-module X, then D is continuous, iff τ_h is a real linear subspace of A.

PROOF : If D is continuous then $A = \tau$. Conversely let τ_h be a linear subspace of A_h. Since $U_{A_h}(\tau_h) \subseteq \tau_h$, τ_h is an ideal of A_h. If A is not unital, adjoin 1 to A and consider X a unital A_1 module. Then $U_{A_1}\tau \subseteq \tau$. Hence if τ_h is linear then it is an ideal of A_h. By the same argument as in Theorem 2.1 (iii) we can see that D is continuous on $\tau_h + i\tau_h$, thus $J = \tau_h + i\tau_h$. Similar argument to Theorem 2.1 (ii), shows that every element in $(A/J)_h$ ($\approx A_h/J_h$ Wright[13]) has finite spectrum, hence $(A/J)_h$ is a semi-simple real Banach Jordan algebra in which every element has non-empty finite spectrum, and by Benslimane and Kaidi[3] it is reduced. Hence there exists idempotents $\overset{0}{e_1}, ..., \overset{0}{e_n} \in A_h/J_h$, such that, $\overset{0}{e_i}\,\overset{0}{e_j} = 0$ $(i \neq j)$ and $U_{\overset{0}{e_i}}(A_h/J_h) = \mathbb{R}\,\overset{0}{e_i}$ $(i = 1, ..., n)$. Moreover $\overset{0}{e_1} + \ldots + \overset{0}{e_n} = \overset{0}{1}$. Let $\{a_m\} \subseteq A_h$ and $a_m \to 0$, then $\overset{0}{a_m} \to 0$ and there exists $\lambda_{im} \in \mathbb{R}$, such that,

$$U_{\overset{0}{e_i}}(\overset{0}{a_m}) = \lambda_{im}\,\overset{0}{e_i}. \qquad \ldots (*)$$

Hence $\lambda_{im} \to 0$ as $m \to \infty$. By (*) $U_{e_i}(a_m) - \lambda_{im} e_i \in J$ for $i = 1, ..., n$. Since $D|_J$ is continuous $D(U_{e_i}(a_m) - \lambda_{im} e_i) \to 0$ as $m \to \infty$. Therefore $D\, U_{e_i}(a_m) \to 0$ for $i = 1, ..., n$ and $e_i \in J$, for each $1 \leq i \leq n$. It follows that $e_1 + ... + e_n \in J$, and $J = A$. ∎

Corollary 2.3 — Let A be a JB^*-algebra then

(i) Every derivation $D : A \to A$ is continuous.

(ii) Every derivation $D : A \to A^*$ is continuous.

PROOF : (i) Consider $D : A \to A$. Let $\tau_h = \{a \in A_h : U_a \sigma = \{0\}\}$. By Corollary 1.5 we have $\tau_h = \sigma_h{}^{\perp}$. Hence τ_h is a linear subspace of A_h and by Theorem 2.2, D is continuous.

(ii) If $D : A \to A^*$ is a derivation, Theorem 1.6 shows that τ is a linear subspace of A and by Theorem 2.2, we have the result. ∎

Corollary 2.3(i) is a well-known result, but it seems that this is a new proof for it.

Theorem 2.4 — Let A be a commutative C^*-algebra, and let X be a Banach Jordan A-module. If $D : A \to X$ is a derivation, then D is continuous.

PROOF : We show that $\tau_h = J_h$. Let $a \in \tau_h$ and $a = a^*$. We have $a^2 A = aAa = U_a A \subseteq \tau$. Hence $a^2 A \subseteq \tau$. Since J is the largest ideal of A, contained in τ (Theorem 2.1), then $a^2 A \subseteq J$. Thus $a^4 \in J$ and since $a = a^*$, we have $a \in J$. ∎

Let A be an associative algebra with minimal left ideals, the smallest left ideal of A, containing all of them is called the left socle of A. The right socle is similarly defined. If A has both minimal left and minimal right ideals and the left socle coincides with the right socle, then it is called the socle of A and is denoted by soc (A). In this case we say for brevity that soc (A) exists. If soc (A) exists then it is an ideal[4]. If A is semiprime with minimal left ideals, then soc (A) exists[4].

Lemma 2.5 — Suppose that A is JB^*-algebra and $D : A \to X$ is a derivation of A to a Banach Jordan A-module. If $e \in A$, is an idempotent, such that $U_e A = \mathbb{C}\, e$, then $e \in \tau$. In the case that A is a C^*-algebra, any linear combination of minimal idempotents lies in τ.

PROOF : Since $U_e A = \mathbb{C} e$, there exists $f \in A^*$, such that $U_e(a) = f(a)\, e\, (a \in A)$. Hence DU_e is continuous and $e \in \tau$. If A is a C^*-algebra and $e_1, ..., e_n$ are minimal idempotents in A, by the first part $e_i \in \tau$ for $i = 1, ..., n$. By § 31 of Bonsall and Duncan[4], dim $e_i A e_j \leq 1$ $(i, j = 1, ..., n)$. Hence for each pair (i, j), with $e_i A e_j \neq \{0\}$ there exists a non-zero $t_{ij} \in A$ such that $e_i A e_j = \mathbb{C}\, t_{ij}$. Let $\{a_k\} \subseteq A$ and $a_k \to 0$. For each $k \in N$ and there exists $\mu_{ijk} \in \mathbb{C}$ such that, $a_k = \mu_{ijk}\, t_{ij}$ $(i, j = 1, ..., n)$. Let $\lambda_1, ..., \lambda_n \in \mathbb{C}$ then :

$$D(U_{\lambda_1 e_1 + \dots + \lambda_n e_n}(a_k)) = D\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \mu_{ijk} t_{ij}\right).$$

But $\mu_{ijk} \to 0$ as $k \to \infty$. Hence $D\ (U_{\lambda_1 e_1 + ... \lambda_n e_n}(a_k)) \to 0$. Therefore $\lambda_1 e_1 + \dots + \lambda_n e_n \in \tau$. ■

Theorem 2.6 — Let A be a C^*-algebra with minimal idempotents, and let X be a Banach Jordan A-module. Then for each derivation $D : A \to X, D$ is continuous on $\overline{\mathrm{soc}(A)}$.

PROOF : By Bonsall and Duncan[4] (30.10), soc (A) exists. Let M denote the set of minimal idempotents of A, then soc $(A) = \sum_{e \in M} eA = \sum_{e \in M} Ae$. Thus for each $a \in \mathrm{soc}\,(A)_h$, there exists $b_1, ..., b_n \in A$ and $e_1, ..., e_n \in M$, such that $a = b_1 e_1 + ... + b_n e_n$. Hence $a^* = e_1 b_1^* + \dots + e_n b_n^* = a$. Thus for each $b \in A$

$$U_a(b) = \sum_{i=1}^{n} \sum_{j=1}^{n} e_i b_i^* b\, b_j e_j.$$

The method used in Lemma 2.5 shows that $a \in \tau$, and by closedness of τ $(\overline{\mathrm{soc}(A)})_h \subseteq \tau$. The same argument as in Theorem 2.1 (iii), shows that D is continuous on $\overline{\mathrm{soc}(A)}$.

Corollary 2.7 — If A is a C^*-algebra and $\overline{\mathrm{soc}(A)} = A$, then any derivation of A to a Banach Jordan A-module is continuous.

Corollary 2.8 — If $A = K(H)$, the C^*-algebra of compact operators on a Hilbert space H, then any derivation of A to a Banach Jordan A-module, is continuous.

PROOF : $A = \overline{\mathrm{soc}(A)}$. ■

3. Discontinuous Derivations

We know that if in a C^*-algebra A, every self-adjoint element has finite spectrum, then A is finite dimensional, but there do exist infinite dimensional JB^*-algebras, in which every self-adjoint element has finite spectrum. To study these algebras we need to study an interesting class of JB-algebras, called spin factors.

Definition 3.1 (Hanche-Olsen and Stormer[5]) — Let H be a real Hilbert space of dimension at least 2, and let $A = H \oplus \mathbb{R}1$ have the norm, $\| t+\lambda \| = \| t \| + |\lambda|$ $(t \in H,\ \lambda \in \mathbb{R})$. Define a product on A as

$$(t+\lambda 1)\circ(u+\mu 1) = \mu a + \lambda b + (\langle t, u \rangle + \lambda\mu)1 \qquad (t, u \in H,\ \lambda, \mu \in \mathbb{R})$$

where $\langle\ ,\ \rangle$ denote the inner product in H. Then A is called a spin factor.

The Spin factor $H \oplus \mathbb{R}1$, is a JC algebra, and for each $t+\lambda 1$ in this algebra $sp(t+\lambda 1) = \{\lambda - \| t \|,\ \lambda + \| t \|\}$ Hanche-Olsen and Stormer[5]. Although it may happen that H is infinite dimensional, but each element in $H \oplus \mathbb{R}1$ has finite spectrum. Results of Benslimane and Kaidi[3] imply that, if A is a JB-algebra in which every element has finite spectrum, then $A = A_1 \oplus \ldots \oplus A_n$, where for each $i = 1, \ldots, n$, A_i is a minimal ideal of A, which is finite dimensional or is an infinite dimensional spin factor. Obviously the complexification of these algebras are infinite dimensional JB^*-algebras, in which every self-adjoint element has finite spectrum. In this stage we are going to show that for these JB^*-algebras there do exist, discontinuous derivations to certain Banach Jordan modules. Let $A = H \oplus \mathbb{R}1$ be a spin factor and X an arbitrary Banach space. Define a bilinear mapping of $A \times X$ to X as

$$(t+\mu, x) \to (t+\mu)\circ x = \mu x \qquad (t \in H,\ \mu \in \mathbb{R},\ \ x \in X).$$

Hence X would be a Banach Jordan A-module, such that $H \circ X = \{0\}$. X is unital, that is, $1 \circ x = x$ for all $x \in X$. This leads us to the following definition

Definition 3.2 — Let $A = H \oplus \mathbb{R}1$ be a spin factor and let X be a unital Jordan A-module. If $H \circ X = \{0\}$, then we call X a degenerate A-module, otherwise X is said to be non-degenerate. X is called completely non-degenerate if X contains no nonzero degenerate submodule, i.e. if $S \subseteq X$ is a submodule and $H \circ S = \{0\}$ then $S = \{0\}$.

Now let X be a degenerate Banach Jordan module for a spin factor $A = H \oplus \mathbb{R}1$. Then any linear operator $T : A \to X$, with $T(1) = 0$ is a derivation. Hence if H is infinite dimensional, then we may find discontinuous derivations of A to degenerate Banach Jordan A-modules.

Theorem 3.3 — Suppose that $A = H \oplus \mathbb{R}1$ is an infinite dimensional spin factor, and X is a unital Banach Jordan A-module. Then every derivation of A to X is continuous iff X is completely non-degenerate.

Proof : Let $S \subseteq X$ is a submodule, such that $H \circ S = \{0\}$, then we may find a disconitnuous operator $T : H \to S$. Extend T on A by defining $T(1) = 0$, then T is a discontinuous derivation of A to X. Conversely let X be completely non-degenerate, and $D : A \to X$ a derivation. Any idempotent in A, is of form $\frac{1}{2}(1 \pm t)$ where t is a unit vector in H, and for each idempotent $e \in A$, we have $U_e A = \mathbb{R}e$ Hanche-Olsen and Stormer[5]. Hence by using the same notation as before $e \in \tau$ for each idempotent $e \in A$ (Lemma 2.5). Let $t \in H$ and $\| t \| = \frac{1}{2}$. Take $e_1 = \frac{1}{2} + t$ and $e_2 = \frac{1}{2} - t$, then $e_1 \circ e_2 = 0$ and $R_{e_1} R_{e_2} = R_{e_2} R_{e_1}$ on $A \oplus X$, since $e_1, e_2 \in C_1(t)$, and $C_1(t)$ is strongly associative in $A \oplus X$. Now for each $x \in \sigma$ we have

$$U_1(x) = U_{e_1+e_2}(x) = 2\,[R_{e_1} R_{e_2} + R_{e_2} R_{e_1} - R_{e_1 \circ e_2}]\,(x) + U_{e_1}(x) + U_{e_2}(x)$$

$$= 4\, R_{e_1} R_{e_2}(x)$$

$$= 2\; e_1 \circ x - 2\, U_{e_1}(x)$$

$$= 2\, e_1 \circ x.$$

Similarly $U_1(x) = 2e_2 \circ x$. Hence $e_1 \circ x = e_2 \circ x$, and it follows that $t \circ x = 0$ for each $x \in \sigma$, and each $t \in H$, with $\| t \| = \frac{1}{2}$. This implies that $H \circ \sigma = \{0\}$ and since X is completely non-degenerate, D is continuous.

Corollary 3.4 — Let A be JB^*-algebra in which every self-adjoint element has finite spectrum, and let X be a Banach Jordan A-module. Then every derivation of A to X is continuous iff X is a completely non-degenerate module for each of the finite dimensional direct summands of A_h.

The above results also have an interesting corollary. Sinclair[10] proved that a homomorphism θ of a C^*-algebra A to a Banach algebra is continuous, iff θ is continuous on each C^*-subalgebra of A generated by a single self-adjoint element a. But the results of this section shows that it is not true for JB^*-algebras. Since as we proved in Theorem 2.4, every derivation of a JB^*-algebra is continuous on $C^*(a)$, for any self-adjoint element a. But if D is a discontinuous derivation of a spin factor to a degenerate Banach Jordan module X, then the homomorphism $\theta : A \to A \oplus X$ defined by $\theta(a) = a + Da$ is discontinuous, although it is continuous on $C(a)$ for each $a \in A$. (Note that by the same notation as in section 2, $\| a + x \|_0 \leq \| a \| \cdot \| x \|$, and $A \oplus X$ is a Banach Jordan algebra.)

REFERENCES

1. E. M. Alfsen, F. W. Shultz, E. Stormer and A. Gelfand, *Advances in Mathematics* **28** (1978), 11-56.
2. M. Bataglia, *Math. Proc. Camb. Phil. Soc.* (1990), 108-317.
3. M. Benslimane and A. Kaidi, *J. Algebra* **113** (1988), 201-206.
4. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Spinger Verlag, 1973.
5. H. Hanche-Olsen and E. Stormer, *Jordan Operator Algebras*, Pitman, 1984.
6. N. Jacobson, Structure and representation of Jordan algebras, *Amer. Math. Soc. Colloq. Publications*, Vol. 39 (1968).

7. B. E. Johnson and A. M. Sinclair, *Am. J. Math.* **90** (1968), 1067-73.
8. G. K. Pedersen, *C*-algebra and their Automorphism Groups*, Academic Press, Inc., New York, 1979.
9. J. R. Ringrose, *J. London Math. Soc.* (2) **5** (1972), 432-38.
10. A. M. Sinclair, *Proc. London Math. Soc.* (3) **29** (1975), 435-52.
11. A. M. Sinclair, *London Math. Soc. Lecture Notes Series* 21, Camb. Univ. Press, 1976.
12. A. Villena, *Derivation on Jordan Banach algebras*, Preprint.
13. J. D. M. Wright. *Mich. Math. J.* **24** (1977), 291-302.
14. E. I. Zelmanov, *Algebra i Logika* **17** (1978), 693-704.