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A GENERAL FIXED POINT THEOREM FOR INVOLUTIONS

J. GÓRNICKI[1] AND B. E. RHOADES[2]

[1]*Department of Mathematics, Technical University of Rzeszów, W. Pola* 2, *P. O. Box* 85, 35-959 *Rzeszów, Poland*

[2]*Department of Mathematics, Indiana University, Bloomington, Indiana* 47405-5701, *U.S.A.*

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We establish a general fixed point principle which includes all known fixed point theorems for involutions as special cases.

The literature contains many fixed point theorems in which the contractive definition is strong enough to guarantee the existence and uniqueness of a fixed point, and for which the fixed point can be obtained by function iteration. If the contractive definition is weak enough, however, then additional restrictions either on the function or on the space, or both, are necessary in order to obtain a fixed point. The main additional assumption considered in this paper is that of requiring that the map T be an involution; i.e., $T^2 = I$, I the identity map on the space.

We shall first establish a general principle for obtaining fixed points, and then use it to obtain known fixed point theorems for involutions as special cases. We then look at situations in which the contractive definition involves a continuous function F, which reduces to known conditions when $F = I$.

Theorem 1 — Let C be a nonempty closed convex subset of a Banach space X, T a selfmap of C satisfying the condition that, for some $x_0 \in C$, there exists a constant c, $0 \leq c < 1$ such that

$$\| x_{n+2} - x_{n+1} \| \leq c \, \| x_{n+1} - x_n \| , \quad n = 0, 1, 2, \ldots , \qquad \ldots (1)$$

where

$$x_{n+1} := \frac{1}{2} (x_n + Tx_n). \qquad \ldots (2)$$

Then $\{x_n\}$ converges to a point p in C. If, in addition, there exist nonnegative constants $\alpha, \beta, \gamma, \delta$, $0 \leq \gamma < 1$, such that

$$\| Tx_n - Tp \| \leq \alpha \| x_n - p \| + \beta \| x_n - Tx_n \| + \gamma \max \{\| p - Tp \|, \| x_n - Tp \|\} + \delta \| p - Tx_n \| \quad \dots (3)$$

for all n sufficiently large, then p is a fixed point of T.

PROOF : Let $x_0 \in C$ such that (1) is satisfied. Then, from (1), it follows that

$$\| x_{n+1} - x_n \| \leq c^n \| x_1 - x_0 \| \quad \dots (4)$$

and $\{x_n\}$ converges to a point p in C.

Suppose that (3) is satisfied. It follows from (2) that $x_n - Tx_n = 2(x_n - x_{n+1})$. Using (4) we obtain $\| x_n - Tx_n \| \leq 2c^n \| x_1 - x_0 \|$, and $\lim \| x_n - Tx_n \| = 0$. Since $\lim x_n = p$, we have $\lim Tx_n = p$.

Taking the limit of (3) as $n \to \infty$ yields $\| p - Tp \| \leq \gamma \| p - Tp \|$, which implies that $p = Tp$.

Corollary 1 (Ćirić[2], Theorem 1) — Let C be a closed convex subset of a Banach space X and let T be a selfmap of C satisfying :

(i) $$\| Tx - Ty \| \leq k \max \{\| x - y \|, \frac{1}{2} \| x - Tx \|, \frac{1}{2} \| y - Ty \|, \frac{1}{2} \| x - Ty \|, \frac{1}{2} \| y - Tx \|\}$$

for all x, y in C and $0 \leq k < 2$;

(ii) $$\left(\frac{k}{2}\right)^{\alpha} \| x - y \| \leq \| T^2 x - y \| \leq \left(\frac{k}{2}\right)^{\beta} \| x - y \|$$

for any $x \in C$ and $y \in \{Fx, Tx, TFx\}$, where $Fx := \frac{1}{2}(x + Tx)$, and $0 \leq \beta \leq \alpha < 1$.

Then T has at least one fixed point.

PROOF : To prove Corollary 1 it is only necessary to verify that conditions (1) and (3) of Theorem 1 are satisfied.

From the definition of F we have

$$\| x - Tx \| = 2 \left\| x - \frac{1}{2}(x + Tx) \right\| = 2 \| x - Fx \| \quad \dots (5)$$

and

$$\| Tx - Fx \| = \left\| Tx - \frac{1}{2}(x + Tx) \right\| = \left\| \frac{1}{2}(x + Tx) - x \right\| = \| Fx - x \|. \quad \dots (6)$$

Put $u = 2(Fx - TFx) + TFx = 2\frac{1}{2}(x + Tx) - TFx = (x - TFx) + Tx$. Then, using (5), we have

$$\| u - TFx \| = \| 2(Fx - TFx) \| = 2 \| Fx - TFx \| = 4 \| Fx - F^2 x \|.$$

Since

$$\| u - TFx \| \leq \| u - Tx \| + \| Tx - TFx \| = \| (x - TFx) + Tx - Tx \| + \| Tx - TFx \|$$

$$= \| x - TFx \| + \| Tx - TFx \| \leq 2 \max \{\| x - TFx \| , \| Tx - TFx \|\},$$

we obtain

$$\| Fx - F^2 x \| = \frac{1}{4} \| u - TFx \| \leq \frac{1}{2} \max \{\| x - TFx \|, \| Tx - TFx \|\}. \quad \dots (7)$$

For each x such that $\| x - TFx \| \leq \| Tx - TFx \|$, from (i),

$$\| Tx - TFx \| \leq k \max \left\{ \| x - Fx \|, \frac{1}{2} \| x - Tx \|, \frac{1}{2} \| Fx - TFx \|, \frac{1}{2} \| x - TFx \| , \frac{1}{2} \| Fx - Tx \| \right\}. \quad \dots (8)$$

Using (5), (6), and (8) and noting that $k < 2$,

$$\| Tx - TFx \| \leq k \max \{\| x - Fx \|, \| Fx - F^2x \|\}. \quad \dots (9)$$

Therefore (9), (8) and (7) imply

$$\| Fx - F^2x \| \leq \frac{1}{2} \| Tx - TFx \| \leq \frac{k}{2} \max \{\| x - Fx \| , \| Fx - F^2x \|\}. \quad \dots (10)$$

For each x such that $\| Tx - TFx \| < \| x - TFx \|$, ... (11)
using (ii) and (i),

$$\| x - TFx \| \leq \left(\frac{k}{2} \right)^{-\alpha} \| T^2 x - TFx \|$$

$$\leq \left(\frac{k}{2} \right)^{-\alpha} \max \left\{\| Tx - Fx \|, \frac{1}{2} \| Tx - T^2x \|, \frac{1}{2} \| Fx - TFx \|, \frac{1}{2} \| Tx - TFx \|, \frac{1}{2} \| Fx - T^2x \|\right\}$$

$$\leq 2 \left(\frac{k}{2} \right)^{-\alpha} \max \left\{\| Tx - Fx \|, \frac{1}{2} \left(\frac{k}{2} \right)^{\beta} \| Tx - x \|, \frac{1}{2} \| Fx - TFx \|, \frac{1}{2} \| Tx - TFx \|, \frac{1}{2} \left(\frac{k}{2} \right)^{\beta} \| Fx - x \|\right\}.$$

Using (6), (5), (11) and noting that $\left(\frac{k}{2} \right)^{1-\alpha} < 1$ and $\left(\frac{k}{2} \right)^{\beta} \leq 1$, we obtain

$$\| x - TFx \| \le 2\left(\frac{k}{2}\right)^{1-\alpha} \max \left\{ \| x - Fx \|, \left(\frac{k}{2}\right)^{\beta} \| x - Fx \|, \right.$$

$$\left. \| Fx - F^2x \|, \frac{1}{2}\left(\frac{k}{2}\right)^{\beta} \| x - Fx \| \right\}$$

$$\le 2\left(\frac{k}{2}\right)^{1-\alpha} \max \{ \| x - Fx \|, \| Fx - F^2x \| \}.$$

Hence, by (7) and (11) one has

$$\| Fx - F^2x \| \le \frac{1}{2} \| x - TFx \| \le \left(\frac{k}{2}\right)^{1-\alpha} \max \{ \| x - Fx \|, \| Fx - F^2x \| \}. \quad \dots (12)$$

Therefore, from (10) and (12), (7) implies that

$$\| Fx - F^2x \| \le \left(\frac{k}{2}\right)^{1-\alpha} \max \{ \| x - Fx \|, \| Fx - F^2x \| \} \quad \dots (13)$$

since $\left(\frac{k}{2}\right) \le \left(\frac{k}{2}\right)^{1-\alpha}$

which implies that

$$\| Fx - F^2x \| \le \left(\frac{k}{2}\right)^{1-\alpha} \| x - Fx \|. \quad \dots (14)$$

Since $0 \le \left(\frac{k}{2}\right)^{1-\alpha} < 1$, (14) implies condition (1) for the sequence $x_n = F^n x$. Then $\{x_n\}$ converges to a point p in C.

Using (i) with $x = x_n$, $y = p$, one obtains

$$\| Tx_n - Tp \| \le k \max \left\{ \| x_n - p \|, \frac{1}{2} \| x_n - Tx_n \|, \right.$$

$$\left. \frac{1}{2} \| p - Tp \|, \frac{1}{2} \| x_n - Tp \|, \frac{1}{2} \| p - Tx_n \| \right\}$$

$$\le k \| x_n - p \| + \frac{k}{2} \| x_n - Tx_n \|$$

$$+ \frac{k}{2} \max \{ \| p - Tp \|, \| x_n - Tp \| \} + \frac{k}{2} \| p - Tx_n \|$$

and (3) is satisfied.

Corollary 1 contains as special cases Theorem 2.1. of Khan and Imdad[8], Theorem 1 of Goebel and Zlotkiewicz[4], the theorem in Iseki[6] (amended to have the map from a closed convex subset of X into itself), and the theorem of Iseki[5].

Rehman and Ahmad[10] established the following result.

Theorem RA — Let C denote a closed convex subset of a complete metrizable linear topological space. Let T be a selfmap of C satisfying

(i) $T^2 = I$ and

(ii) $q(Tx, -Ty) \leq \frac{\alpha}{2} \max\left\{ q(x-y), \frac{1}{2} q(x-Tx), \frac{1}{2} q(y-Ty), \frac{1}{3} q(x-Ty), \frac{1}{3} q(y-Tx) \right\}$

for all $x, y \in C$, where $0 \leq \alpha \leq 1$. Then T has at least one fixed point.

Except for the choice of metric, this result is also a special case of Corollary 1.

For uniformly convex spaces we have the following result, which contains Theorem 2 of Goebel and Zlotkiewicz[4] and Theorem 2.4 of Khan and Imdad[8] as special cases.

In the course of our proof we will make use of the following Lemma due to Goebel and Zlotkiewicz[4].

Lemma GZ — If B is a uniformly convex Banach space and if x, z, $u \in B$ satisfy the conditions

$$\| z - x \| \leq R, \quad \| u - x \| \leq R, \quad \text{and} \quad \left\| \frac{u+z}{2} - x \right\| \geq r$$

then

$$\| u - z \| \leq R\delta^{-1}\left(\frac{R-r}{R}\right).$$

Theorem 2 — Let B be a uniformly convex Banach space, C a closed and convex subset of B, T a selfmap of C satisfying conditions (i) $T^2 = I$ and condition (ii) of Corollary 1, with k such that $k\delta^{-1}(1-(1/k)) < 4$. Then T has at least one fixed point.

PROOF : Using the technique in Khan and Imdad[8] (due to Goebel and Zlotkiewicz[4]), define $G = \frac{1}{2}(I+T), y = Gx, z = Ty$ and $u = 2y - z$. Then, by (ii) we have

$$\| z - x \| = \| Ty - T^2x \| \leq k \max \left\{ \| y - Tx \|, \frac{1}{2} \| y - Ty \|, \right.$$

$$\left. \frac{1}{2} \| Tx - T^2x \|, \frac{1}{2} \| y - T^2x \|, \| Tx - Ty \| \right\}. \quad \ldots (15)$$

Using (i) $\| y - Tx \| = \| Gx - Tx \| = \frac{1}{2} \| x - Tx \|$ and $\| y - T^2x \| = \| Gx - x \|$ $= \frac{1}{2} \| x - Tx \|$.

From (ii)

$$\|Tx-Ty\|$$
$$\leq k\max\left\{\|x-y\|,\frac{1}{2}\|x-Tx\|,\frac{1}{2}\|y-Ty\|,\frac{1}{2}\|x-Ty\|,\frac{1}{2}\|y-Tx\|\right\} \quad \dots (16)$$

and, substituting (16) into (15) we can ignore each term on the right hand side of (16) except $\frac{1}{2}\|x-Ty\|=\frac{1}{2}\|T^2x-Ty\|$, and we obtain

$$\|z-x\|=\|Ty-T^2x\|$$
$$\leq k\max\left\{\frac{1}{2}\|x-Tx\|,\frac{1}{2}\|y-Ty\|,\frac{k}{2}\|T^2x-Ty\|\right\}$$
$$=\frac{k}{2}\max\{\|x-Tx\|,\|y-Ty\|\}=M,\text{ say.} \quad \dots (17)$$

From (16)

$$\|u-x\|=\|2y-z-x\|=\|Tx-Ty\|$$
$$\leq k\max\left\{\frac{1}{2}\|x-Tx\|,\frac{1}{2}\|y-Ty\|,\frac{1}{2}\|x-Ty\|\right\}.$$

Since $\|x-Ty\|=\|T^2x-Ty\|$, it follows from (15) that

$$\|u-x\|\leq\frac{k}{2}\max\{\|x-Tx\|,\|y-Ty\|\}, \quad \dots (18)$$

inequalities (17) and (18) imply that u and z both lie in the closed ball with radius M. Therefore $\|z-u\|\leq 2M$. But $\|z-u\|=\|Ty-2y-z\|=2\|y-Ty\|$, which implies that $\|y-Ty\|\leq M$, which implies that $\|y-Ty\|\leq\frac{k}{2}\|x-Tx\|$. Hence (17) and (18) imply that $\|z-x\|\leq\frac{k}{2}\|x-Tx\|$ and $\|u-x\|\leq\frac{k}{2}\|x-Tx\|$. Because

$$\left\|\frac{u+z}{2}-x\right\|=\frac{1}{2}\|x-Tx\|,$$

define $R=\frac{k}{2}\|x-Tx\|$ and $r=\frac{1}{2}\|x-Tx\|$. Then, from Lemma GZ,

$$\|u-z\|\leq\frac{k}{2}\delta^{-1}\left(1-\frac{1}{k}\right)\|x-Tx\|,$$

which yields

$$\|G^2x-Gx\|\leq\frac{k}{2}\delta^{-1}\left(1-\frac{1}{k}\right)\|Gx-x\|,$$

which implies (1). It has already been shown that contractive condition (ii) satisfies (3). The result now follows from Theorem 1.

One can omit the hypothesis that $T^2 = I$ by adding the hypotheses that C is bounded and that T^2 satisfies a certain contractive condition. An example of this is the following result, which contains Theorem 3 of Goebel and Zlotkiewicz[4] and Theorem 2.5 of Khan and Imdad[8] as special cases.

Theorem 3 — Let B be a uniformly convex Banach space, C a closed, bounded, convex subset of B, T a selfmap of C satisfying the conditions of Theorem 2, and T^2 is continuous and satisfies

$$\| T^2x - T^2y \| \le a \| x - y \| + b(\| x - T^2 x \| + \| y - T^2y \|) + c(\| x - T^2y \| + \| y - T^2x \|) \qquad \text{... (19)}$$

for all $x, y \in C$, where $a, b, c \ge 0$ and $a + 2b + 2c \le 1$.
Then T has at least one fixed point.

PROOF : By Theorem 2 of Goebel *et al.*[3], T^2 has at least one fixed point in C. It is easy to show that $C^* := \{x \in C : T^2 x = x\}$ is closed and convex. Moreover, $T(C^*) = C^*$, and $T^2 = I$ on C^*. Now apply Theorem 2.

Several authors have used contractive definitions in which the right hand side of the contractive definition has x and y replaced by a function F for which $F^2 = I$, and conclude that T and F have a common coincidence point. We are able to obtain fixed points for such theorems.

Theorem 4 — Let B be a Banach space, C a closed and convex subset of X, T and F selfmaps of C satisfying

(i) $T^2 = F^2 = I$, (ii) T and F commute, and

(iii) $\| Tx - Ty \| \le k \max$

$$\left\{ \| Fx - Fy \|, \frac{1}{2} \| Fx - Tx \|, \frac{1}{2} \| Fy - Ty \|, \frac{1}{2} \| Fx - Ty \|, \frac{1}{2} \| Fy - Tx \| \right\} \qquad \text{... (20)}$$

for all $x, y \in C$, where $0 \le k < 2$. Then there exists at least one common fixed point for T and F in C.

The proof is easy. As in Khan and Imdad[8], set $x = Fx$ and $y = Fy$ in (20) to obtain that TF satisfies condition (ii) of Corollary 1. By Theorem 2, TF has a fixed point p in C; i.e., $TFp = p$. Then using (i) again, one obtains $Fp = Tp$.

Now set $x = Fp$, $y = p$ in (20) to obtain, using the fact that $Tp = Fp$,

$$\| p - Tp \| = \| TFp - Tp \| \le k \max \left\{ \| F^2 p - Fp \|, \frac{1}{2} \| F^2p - TFp \|, \frac{1}{2} \| Fp - Tp \|, \frac{1}{2} \| F^2p - Tp \|, \frac{1}{2} \| Fp - TFp \| \right\}$$

$$\le k \| p - Tp \|,$$

from which it follows that p is a common fixed point for T and F.

Theorem 2.6 of Khan and Imdad[8] is a special case of Theorem 4. Except for the change in the metric, Theorem 3 of Rehman and Ahmad[10] is also a special case of Theorem 4.

For uniformly convex spaces we have the following results.

Theorem 5 — Let B be a uniformly convex space, C a closed and convex subset of B. Let T and F be selfmaps of C satisfying (20) with $k\delta^{-1}(1-(1/k))<4$, (i) $T^2=F^2=I$, and T and F commute. Then T and F have a common fixed point in C.

PROOF : Setting $x = Fx$, $y = Fy$ in (20) it follows from Theorem 2 that TF has a fixed point in C. Then (i), (ii) imply that the fixed point of TF is a coincidence point for T and F. As in the proof of Theorem 4 it is then possible to show that the coincidence point for T and F is also a common fixed point.

Theorem 2.6 of Khan and Imdad[8] and Theorem 2.3 of Khan[7] are special cases of Theorem 5.

Theorem 6 — Let B be a uniformly convex space, C a closed, bounded and convex subset of B. Let T and F be selfmaps of C satisfying (20) with $k\delta^{-1}(1-(1/k))<4$, (i) $F^2=I$, (ii) T and F commute, and T^2 is continuous and satisfies (19). Then T and F have a common fixed point.

PROOF : Since T^2 satisfies (19), Theorem 2 of Goebel *et al.* applies, and T^2 has at least one fixed point in C. The set C^* is closed convex. Since T and F commute, F is also a selfmap of C^*. Also, $T^2 = I$ on C^*. Theorem 5 then applies to C^*.

Theorem 2.7 of Khan and Imdad[8] and Theorem 2.3 of Khan[7] are special cases of Theorem 6.

The following appears in Khan[7].

Theorem K : (Theorem 3.1) — Let X be a Banach space and $x_0 \in X$ be arbitrary. Let F and G be selfmaps of X such that the following are true :

(a) F and G commute,

(b) $F(X) \subset G(X)$,

(c) G is continuous and linear,

(d) for all $x, y \in X$,

$$\| Fx - Fy \| \le a \| Gx-Gy \| + b \{\| Gx - Fx \| + \| Gy - Fy \|\} + c \{\| Gx - Fy \| + \| Gy - Fx \|\}, \quad \ldots (21)$$

where $a, b, c \ge 0, a+2b+2c \le 1$.

Let $x_0 \in X$ be arbitrary. If $\{x_n\}$ is a sequence in X satisfying $Gx_{n+1} = \frac{1}{2}(Gx_n + Fx_n)$, $n = 0, 1, \ldots$, and for which $\{Gx_n\}$ converges, then there is at least one common fixed point of F and G.

By adding a few restrictions we are able to prove a stronger result.

Theorem 7 — Let X be a uniformly convex Banach space, F and G selfmaps of X such that the following are true :

(a) FG is continuous,

(b) $F(X) \subset G(X)$,

(c) $F^2 = G^2 = I$,

(d) F and G satisfy (21) with $b > 0$.

Then F and G have a unique common fixed point.

PROOF : Set $x = Gx$, $y = Gy$ in (21) to obtain the result that FG satisfies the hypotheses of the Theorem of Goebel *et al.*[3]. Thus there exists a point p in X with $FGp = p$. It then follows that p is a coincidence point of F and G.

Now set $x = Gp$, $y = p$ in (21), and use the fact that $Fp = Gp$ to obtain the inequality $\| p - Fp \| \leq (a + 2c) \| p - Fp \|$, which implies that $p = Fp$, and hence that $p = Gp$. The uniqueness follows from (21).

Theorem 8 — Let C be a nonempty closed convex subset of a Banach space X. Let F and G be selfmaps of C satisfying the following conditions :

(i) $F^2 = G^2 = I$,

(ii) for all $x, y \in C$ there exists a constant h, $0 \leq h < 1$, such that

$$\| Fx - Fy \| \leq h \max \{\| Gx - Gy \|, \| Gx - Fx \|, \| Gy - Fy \|, \| Gx - Fy \|, \| Gy - Fx \|\}. \quad \text{... (22)}$$

Then F and G have a unique common fixed point.

PROOF : Set $x = Gx$, $y = Gy$ in (22) to get that FG satisfies the conditions of Theorem 1 of Ciric[1], and hence has a unique fixed point p. It then follows that p is a coincidence point of F and G.

Now set $x = Gp$, $y = p$ in (22) to obtain $\| p - Fp \leq k \| p - Fp \|$, which implies that $p = Fp$ and hence that $p = Gp$. Uniqueness follows from (22).

Theorem 1 of Pathak[9] is a special case of Theorem 8. To see this, the contractive definition in Pathak is

$$\| Fx - Fy \|^2 \leq q \max \{\| Gx - Fx \| \| Gy - Fy \|, \| Gx - Fy \| \| Gy - Fx \|, \| Gx - Fx \| \| Gy - Fx \|, \| Gx - Fx \| \| Gy - Fy \|\}, \quad \text{... (23)}$$

where $q \in (0, 1)$.

Condition (23) implies

$$\| Fx - Fy \|^2 \leq q \max \{\| Gx - Fx \|^2, \| Gy - Fy \|^2, \| Gx - Fy \|^2, \| Gy - Fx \|^2\},$$

or,

$$\| Fx - Fy \| \leq h \max \{ \| Gx - Fx \|, \| Gy - Fy \|, \| Gx - Fy \|, \| Gy - Fx \|\},$$

where $h = \sqrt{q}$. Then (23) is a special case of (22). Pathak[9] also assumes other additional conditions that are not needed.

It is well known that, without additional assumptions, either on the space or on S and T, one of the most general contractive definitions which yields a unique common fixed points is the following : for each $x, y \in X$, there exists a constant k, $0 \leq k < 1$ such that

$$\| Sx, Ty \| \leq k \max \{\| x - y \|, \| x - Sx \|, \| y - Ty \|, [\| x - Ty \| + \| y - Sx \|]/2\}.$$

A natural question is whether or not adding the hypothesis that F and G are involutions would allow one to prove a fixed point theorem with $[\| x - Ty \| + \| y - Sx \|]/2$ replaced by $\max \{\| x - Ty \|, \| y - Sx \|\}$. The following result show that such an attempt forces S and T to be the identity map.

Theorem 9 — Let (X, d) be a complete metric space, $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for each $t > 0$, S, T selfmaps of X satisfying : for each $x, y \in X$,

$$d(Sx, Ty) \leq \phi(M(x, y)), \quad \ldots (24)$$

where

$$M(x, y) = \text{Max } \{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\},$$

and $S^2 = T^2 = I$. Then $S = T = I$.

PROOF : Let $x \in X$. There are three possibilities : (a) $x = Sx$, (b) $x = Tx$, or (c) $x \neq Sx$ and $x \neq Tx$.

If (a) holds, then, using (24) with $y = x$, we obtain

$$d(x, Tx) = d(Sx, Tx) \leq \phi(\max \{d(x, x), d(x, Sx), d(x, Tx)\}) = \phi(d(x, Tx)).$$

If $x \neq Tx$, we obtain $d(x, Tx) < d(x, Tx)$, a contradiction. Therefore $x = Tx$.

Similarly, condition (b) also implies that $x = Sx$.

If (c) holds, then, using (24) with $y = Tx$,

$$d(Sx, T^2x) \leq \phi(\max \{d(x, Tx), d(x, Sx), d(Tx, T^2x), d(x, T^2x), d(Tx, Sx)\}).$$

But $\quad d(Sx, Tx) \leq \phi(\max \{d(x, Sx), d(x, Tx)\}).$

Therefore

$$d(Sx, x) \leq \phi(\max \{d(x, Sx), d(x, Tx)\}) < \max \{d(x, Sx), d(x, Tx)\},$$

and $d(Sx, x) < d(Tx, x)$.

Now set $x = Sx$, $y = x$ in (24) to get

$$d(S^2x, Tx) \leq \phi(\max \{d(Sx, x), d(Sx, S^2x), d(x, Tx), d(Sx, Tx), d(x, T^2x)\})$$
$$= \phi(\max \{d(x, Sx), d(x, Tx)\}),$$

which implies that $d(x, Tx) < d(x, Sx)$. Thus we have $d(Sx, x) < d(Tx, x) < d(Sx, x)$, a contradiction. Therefore $x = Sx$ and hence $x = Tx$, so that S and T are the identity map on X.

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