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FUZZY DEDUCTIVE SYSTEMS OF HILBERT ALGEBRAS

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In this paper we define and discuss the fuzzy deductive systems of Hilbert algebras. We construct a new fuzzy deductive system from old, and study fuzzy relations on Hilbert algebras.

1. INTRODUCTION

The notion of a deductive system of a Hilbert algebra was introduced by Diego[6], and studied further by Busneag[2-4]. The concept of a fuzzy set and a fuzzy relation on a set was defined by Zadeh[9, 10]. Fuzzy relations on a group have been studied by Bhattacharya and Mukherjee[1].

In this paper we consider the fuzzification of deductive systems of Hilbert algebras, and study their properties. We also discuss fuzzy relations on a Hilbert algebra. In particular, we prove that (i) if μ and ν are fuzzy deductive systems of a Hilbert algebra A then $\mu \times \nu$ is a fuzzy deductive system of $A \times A$, (ii) if $\mu \times \nu$ is a fuzzy deductive system of $A \times A$ then either μ or ν is a fuzzy deductive system of A, and (iii) if ν is a fuzzy set in a Hilbert algebra A and μ_ν is the strongest fuzzy relation on A then ν is a fuzzy deductive system of A if and only if μ_ν is a fuzzy deductive system of $A \times A$. An example is given to show that if $\mu \times \nu$ is a fuzzy deductive system of $A \times A$, then μ and ν both need not be fuzzy deductive systems of A.

2. PRELIMINARIES

In this section we include some elementary aspects of Hilbert algebras and fuzzy theories that are necessary for this paper, and for more details we refer to [1]-[6], [8] and [9] under References.

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Definition 2.1 — A Hilbert algebra is a triple $(A, \to, 1)$, where A is a nonempty set, $\to$ is a binary operation on A, $1 \in A$ is an element such that the following three axioms are satisfied for every $x, y, z \in A$:

(i) $x \to (y \to x) = 1$,

(ii) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,

(iii) If $x \to y = y \to x = 1$ then $x = y$.

If A is a Hilbert algebra, then the relation $x \leq y$ iff $x \to y = 1$ is a partial order on A, which will be called the natural ordering on A; with respect to this ordering 1 is the largest element of A.

A Hilbert algebra A satisfies the following property :

(iv) $1 \to x = x$.

Definition 2.2 — If A is a Hilbert algebra, a subset D of A is called a deductive system of A if it satisfies :

(i) $1 \in D$,

(ii) If x, $x \to y \in D$, then $y \in D$.

Definition 2.3 — Let S be a set. A fuzzy set in S is a function $\mu : S \to [0, 1]$.

Definition 2.4 — Let μ be a fuzzy set in a set S. For $\alpha \in [0, 1]$, the set

$$\mu_\alpha = \{x \in S \mid \mu(x) \geq \alpha\}$$

is called a level subset of μ.

Definition 2.5 — A fuzzy relation on any set S is a fuzzy set $\mu : S \times S \to [0, 1]$.

Definition 2.6 — If μ is a fuzzy relation on a set S and ν is a fuzzy set in S, then μ is a fuzzy relation on ν if

$$\mu(x, y) \leq \min \{\nu(x), \nu(y)\}$$

for all $x, y \in S$.

Definition 2.7 — Let μ and ν be fuzzy sets in a set S. The Cartesian product of μ and ν is defined by

$$(\mu \times \nu)(x, y) = \min \{\mu(x), \nu(y)\}$$

for all $x, y \in S$

Lemma 2.8 — Let μ and ν be fuzzy sets in a set S. Then

(i) $\mu \times \nu$ is a fuzzy relation on S,

(ii) $(\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha$ for all $\alpha \in [0, 1]$.

Definition 2.9 — If ν is a fuzzy set in a set S, the strongest fuzzy relation on S that is a fuzzy relation on ν is μ_ν, given by

$$\mu_\nu(x, y) = \min \{\nu(x), \nu(y)\}$$

for all $x, y \in S$.

Lemma 2.10 — For a given fuzzy set ν in a set S, let μ_ν be the strongest fuzzy relation on S. Then for $\alpha \in [0, 1]$, we have that $(\mu_\nu)_\alpha = \nu_\alpha \times \nu_\alpha$.

3. Fuzzy Deductive Systems

Definition 3.1 — Let A be a Hilbert algebra. A fuzzy set μ in A is called a fuzzy deductive system of A if it satisfies :

(i) $\mu(1) \geq \mu(x)$ for all $x \in A$,

(ii) $\mu(y) \geq \min\ \{\mu(x), \mu(x \to y)\}$ for all $x, y \in A$.

Proposition 3.2 — Let μ be a fuzzy deductive system of a Hilbert algebra A. Then

(i) if $\mu(x \to y) = \mu(1)$ then $\mu(x) \leq \mu(y)$.

(ii) if $x \leq y$ then $\mu(x) \leq \mu(y)$.

(iii) if $x \to (y \to z) = 1$ then $\mu(z) \geq \min\ \{\mu(x), \mu(y)\}$.

Proof : (i) If $\mu(x \to y) = \mu(1)$ then

$$\mu(y) \geq \min\ \{\mu(x), \mu(x \to y)\} = \min\ \{\mu(x), \mu(1)\} = \mu(x).$$

(ii) If $x \leq y$ then $x \to y = 1$. Hence by (i) we get (ii).

(iii) If $x \to (y \to z) = 1$ then $x \leq y \to z$. So by (ii), $\mu(x) \leq \mu(y \to z)$. Hence

$$\mu(z) \geq \min\ \{\mu(y), \mu(y \to z)\} \geq \min\ \{\mu(x), \mu(y)\}.$$

Theorem 3.3 — Let μ be a fuzzy set in a Hilbert algebra A. Then μ is a fuzzy deductive system of A if and only if for every $\alpha \in [0, 1]$, the level subset μ_α is a deductive system of A, when $\mu_\alpha \neq \phi$.

Proof : Let μ be a fuzzy deductive system of A. According to Definition 3.1(i), we have $\mu(1) \geq \mu(x)$ for all $x \in A$; in particular, $\mu(1) \geq \mu(x) \geq \alpha$ for every $x \in \mu_\alpha$. Hence $1 \in \mu_\alpha$. Let $x, x \to y \in \mu_\alpha$. Then $\mu(x) \geq \alpha$ and $\mu(x \to y) \geq \alpha$. It follows from Definition 3.1(ii) that

$$\mu(y) \geq \min\ \{\mu(x), \mu(x \to y)\} \geq \alpha,$$

so that $y \in \mu_\alpha$. Therefore μ_α is a deductive system of A.

Conversely we only need to show (i) and (ii) of Definition 3.1 are true. If (i) is false, then there exists $x_0 \in A$ such that $\mu(1) < \mu(x_0)$. Let $\alpha_0 = \frac{1}{2}(\mu(1) + \mu(x_0))$. Then $\mu(1) < \alpha_0$ and $0 \leq \alpha_0 < \mu(x_0) \leq 1$. Hence $x_0 \in \mu_{\alpha_0}$, and $\mu_{\alpha_0} \neq \phi$. Since μ_{α_0} is a deductive system of A, therefore $1 \in \mu_{\alpha_0}$, and so $\mu(1) \geq \alpha_0$. This is a contradiction and (i) of Definition 3.1 is true. Now assume that (ii) of Definition 3.1 is false. Then there exist $x_0, y_0 \in A$ such that

$$\mu(y_0) < \min\ \{\mu(x_0),\ \mu(x_0 \to y_0)\}.$$

Let $\beta_0 = \frac{1}{2}(\mu(y_0) + \min\ \{\mu(x_0), \mu(x_0 \to y_0)\})$. Then $\mu(y_0) < \beta_0$ and $0 \leq \beta_0 < \min\ \{\mu(x_0), \mu(x_0 \to y_0)\} \leq 1$. It follows that $\mu(x_0) > \beta_0$ and $\mu(x_0 \to y_0) > \beta_0$, so that $x_0 \in \mu_{\beta_0}$ and $x_0 \to y_0 \in \mu_{\beta_0}$. This means that $\mu_{\beta_0} \neq \phi$. As μ_{β_0} is a deductive system of A, we have $y_0 \in \mu_{\beta_0}$, and so $\mu(y_0) \geq \beta_0$, a contradiction. This completes the proof.

Definition 3.4 — Let μ be a fuzzy deductive system of a Hilbert algebra A. The deductive systems μ_α, $\alpha \in [0, 1]$, are called level deductive systems of μ, when $\mu_\alpha \neq \phi$.

Theorem 3.5 — Any deductive system of a Hilbert algebra A can be realized as a level deductive system of some fuzzy deductive system of A.

PROOF : Let D be a deductive system of a Hilbert algebra A and let μ be a fuzzy set in A defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D, \\ 0 & \text{if } x \notin D, \end{cases}$$

where α is a fixed number in $(0, 1)$. Note that $1 \in D$, so that $\mu(1) = \alpha \geq \mu(x)$ for all $x \in A$. Let $x, y \in A$. We will divide into the following cases to verify that μ satisfies the condition (ii) of Definition 3.1.

If $x \in D$ and $x \to y \in D$, then $y \in D$. Thus

$$\mu(y) = \mu(x) = \mu(x \to y) = \alpha,$$

and so

$$\mu(y) \geq \min\ \{\mu(x), \mu(x \to y)\}.$$

If $x \notin D$ and $x \to y \notin D$, then $\mu(x) = \mu(x \to y) = 0$. Hence

$$\mu(y) \geq \min\ \{\mu(x), \mu(x \to y)\}.$$

If exactly one of x and $x \to y$ belongs to D, then exactly one of $\mu(x)$ and $\mu(x \to y)$ is equal to 0. Hence

$$\mu(y) \geq \min\ \{\mu(x), \mu(x \to y)\}.$$

The results above show $\mu(y) \geq \min\ \{\mu(x), \mu(x \to y)\}$ for all $x, y \in A$. Therefore μ is a fuzzy deductive system of A and obviously $\mu_\alpha = D$. The proof is complete.

Theorem 3.6 — Let μ be a fuzzy deductive system of a Hilbert algebra A. Then two level deductive systems $\mu_{\alpha_1}, \mu_{\alpha_2}$ (with $\alpha_1 < \alpha_2$) of μ are equal if and only if there is no $x \in A$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.

PROOF : Assume that $\mu_{\alpha_1} = \mu_{\alpha_2}$ for $\alpha_1 < \alpha_2$. If there exists $x \in A$ such that $\alpha_1 \leq \mu(x) < \alpha_2$, then μ_{α_2} is a proper subset of μ_{α_1}. This is impossible. Conversely, suppose that there is no $x \in A$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Note that $\alpha_1 < \alpha_2$ implies $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. If $x \in \mu_{\alpha_1}$, then $\mu(x) \geq \alpha_1$, and so $\mu(x) \geq \alpha_2$ because $\mu(x) \nless \alpha_2$. Hence $x \in \mu_{\alpha_2}$, which says that $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$. Thus $\mu_{\alpha_1} = \mu_{\alpha_2}$. This completes the proof.

Let μ be a fuzzy set in A and let $\mathrm{Im}(\mu)$ denote the image of μ.

Theorem 3.7 — Let μ be a fuzzy deductive system of a Hilbert algebra A. If $\mathrm{Im}(\mu) = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, where $\alpha_1 < \alpha_2 < .... < \alpha_n$, then the family of deductive systems μ_{α_i} $(i = 1, 2, ..., n)$ constitutes all the level deductive systems of μ.

PROOF : Let $\alpha \in [0, 1]$ and $\alpha \notin \mathrm{Im}(\mu)$. If $\alpha < \alpha_1$, then $\mu_{\alpha_1} \subseteq \mu_\alpha$. Since $\mu_{\alpha_1} = A$, we have $\mu_\alpha = A$ and $\mu_\alpha = \mu_{\alpha_1}$. Assume that $\alpha_i < \alpha < \alpha_{i+1}$ $(1 \le i \le n-1)$, then there is no $x \in A$ such that $\alpha \le \mu(x) < \alpha_{i+1}$. It follows from Theorem 3.6 that $\mu_\alpha = \mu_{\alpha_{i+1}}$. This shows that for any $\alpha \in [0, 1]$ with $\alpha \le \mu(1)$, the level deductive system μ_α is in $\{\mu_{\alpha_i} \mid 1 \le i \le n\}$. This completes the proof.

The following lemma is obvious, and we omit the proof.

Lemma 3.8 — Let A be a Hilbert algebra and let μ be a fuzzy deductive system of A. If α and β belong to $\mathrm{Im}(\mu)$ such that $\mu_\alpha = \mu_\beta$, then $\alpha = \beta$.

Theorem 3.9 — Let μ and ν be two fuzzy deductive systems of a Hilbert algebra A such that μ and ν have the finite images, and have the identical family of level deductive systems. If $\mathrm{Im}(\mu) = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $\mathrm{Im}(\nu) = \{\beta_1, \beta_2, ..., \beta_n\}$, where $\alpha_1 > \alpha_2 > \ldots > \alpha_m$ and $\beta_1 > \beta_2, \ldots > \beta_n$, then

(i) $m = n$;

(ii) $\mu_{\alpha_i} = \nu_{\beta_i}$, for $i = 1, 2, ..., m$;

(iii) if $x \in A$ such that $\mu(x) = \alpha_i$ then $\nu(x) = \beta_i$, for $i = 1, 2, ..., m$.

PROOF : Using Theorem 3.7 we have that the only level deductive systems of μ and ν are μ_{α_i} and ν_{β_i}, respectively. Since μ and ν have the identical family of level deductive systems, it follows that $m = n$, and so (i) holds. Using again Theorem 3.7 we get that

$$\{\mu_{\alpha_1}, \mu_{\alpha_2}, ..., \mu_{\alpha_m}\} = \{\nu_{\beta_1}, \nu_{\beta_2}, ..., \nu_{\beta_m}\},$$

and by Theorem 3.6 we have

$$\mu_{\alpha_1} \subset \mu_{\alpha_2} \subset .... \subset \mu_{\alpha_m} = A \text{ and } \nu_{\beta_1} \subset \nu_{\beta_2} \subset .... \subset \nu_{\beta_m} = A.$$

Hence $\mu_{\alpha_i} = \nu_{\beta_i}$, for $i = 1, 2, ..., m$; and (ii) holds.

Let $x \in A$ be such that $\mu(x) = \alpha_i$ and let $\nu(x) = \beta_j$. Then $x \in \mu_{\alpha_i} = \nu_{\beta_i}$, and so $\nu(x) \ge \beta_i$. Hence $\beta_j \ge \beta_i$, which implies $\nu_{\beta_j} \subseteq \nu_{\beta_i}$. Since $x \in \nu_{\beta_j} = \mu_{\alpha_j}$, therefore $\alpha_i = \mu(x) \ge \alpha_j$. It follows that $\mu_{\alpha_i} \subseteq \mu_{\alpha_j}$. By (ii), $\nu_{\beta_i} = \mu_{\alpha_i}$ $\mu_{\alpha_j} = \nu_{\beta_j}$. Consequently $\nu_{\beta_i} = \nu_{\beta_j}$, and by Lemma 3.8 we have $\beta_i = \beta_j$. Thus $\nu(x) = \beta_i$. The proof is complete.

Theorem 3.10 — Let μ and ν be as in Theorem 3.9. Then $\mu = \nu$ if and only if $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$.

PROOF : $(\Rightarrow)$ This is clear.

$(\Leftarrow)$ Suppose that $\mathrm{Im}(\mu) = \mathrm{Im}(\nu) = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, where $\alpha_1 > \alpha_2 > \ldots > \alpha_n$. Let $x_1, ..., x_n$ be distinct elements of A such that $\mu(x_i) = \alpha_i$ $(1 \le i \le n)$. By Theorem 3.9,

$\nu(x_i) = \alpha_i$, $(1 \le i \le n)$. Since for any $x \in A$ there exists some α_i such that $\mu(x) = \alpha_i$, and so $x \in \mu_{\alpha_i} = \nu_{\alpha_i}$. Hence $\nu(x) \ge \alpha_i$, it follows that $\nu(x) \ge \mu(x)$. By the same argument, we have $\mu(x) \ge \nu(x)$. Therefore $\mu(x) = \nu(x)$, showing that $\mu = \nu$. This completes the proof.

Theorem 3.11 — Let A be a Hilbert algebra and let μ be a fuzzy set in A with $\mathrm{Im}(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$, where $\alpha_0 > \alpha_1 > \dots > \alpha_k$. Suppose that there exists a chain of deductive systems of A :

$$D_0 \subset D_1 \subset \dots \subset D_k = A$$

such that $\mu(\overline{D}_n) = \alpha_n$, where $\overline{D}_n = D_n - D_{n-1}$, $D_{-1} = \phi$, for $n = 0, 1, \dots, k$. Then μ is a fuzzy deductive system of A.

PROOF : Since $1 \in D_0$, we have $\mu(1) = \alpha_0 \ge \mu(x)$ for all $x \in A$. In order to prove that μ satisfies the condition (ii) of Definition 3.1, we divide into the following cases:

If x and y belong to the same $\overline{D}_n$, then $\mu(x) = \mu(y) = \alpha_n$, and so

$$\mu(y) \ge \min\{\mu(x), \mu(x \to y)\}.$$

Assume that $x \in \overline{D}_i$ and $y \in \overline{D}_j$ for every $i \ne j$. Without loss of generality, we may assume that $i < j$. Then $\mu(x) = \alpha_i > \alpha_j = \mu(y)$, and so

$$\min\{\mu(y), \mu(y \to x)\} \le \mu(y) < \mu(x).$$

Since $x \in \overline{D}_i$, we have $x \in D_i$. It follows that $x \in D_{j-1}$ as $i \le j-1$. Now we assert that $x \to y \notin D_{j-1}$. In fact, if not, then $x \to y \in D_{j-1}$ and $x \in D_{j-1}$ imply $y \in D_{j-1}$, which contradicts to $y \in \overline{D}_j = D_j - D_{j-1}$. Hence $\mu(x \to y) \le \alpha_j$, and so

$$\mu(y) \ge \min\{\mu(x), \mu(x \to y)\}.$$

Summarizing the above results, we obtain that $\mu(y) \ge \min\{\mu(x), \mu(x \to y)\}$ for all $x, y \in A$. Therefore μ is a fuzzy deductive system of A.

Theorem 3.12 — Let μ be a fuzzy deductive system of a Hilbert algebra A. If $\mathrm{Im}(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$ with $\alpha_0 > \alpha_1 > \dots > \alpha_k$, then $D_n = \mu_{\alpha_n}$, $n = 0, 1, \dots, k$, are deductive systems of A and $\mu(\overline{D}_n) = \alpha_n$, $n = 0, 1, \dots, k$, where $\overline{D}_n = D_n - D_{n-1}$ and $D_{-1} = \phi$.

PROOF : By Theorem 3.7, $D_n = \mu_{\alpha_n}$ $(n = 0, 1, \dots, k)$ is a deductive system of A. Obviously, $\mu(D_0) = \alpha_0$. Since $\mu(D_1) = \{\alpha_0, \alpha_1\}$, for $x \in \overline{D}_1$ we have $\mu(x) = \alpha_1$, namely $\mu(\overline{D}_1) = \alpha_1$. Repeating the above argument, we have $\mu(\overline{D}_n) = \alpha_n$ $(0 \le n \le k)$. This completes the proof.

Theorem 3.13 — If μ is a fuzzy deductive system of a Hilbert algebra A, then the set

$$A_\mu := \{x \in A \mid \mu(x) = \mu(1)\}$$

is a deductive system of A.

PROOF : Clearly $1 \in A_\mu$. Assume that $x \in A_\mu$ and $x \rightarrow y \in A_\mu$. Then $\mu(x) = \mu(1) = \mu(x \rightarrow y)$. Since μ is a fuzzy deductive system of A, therefore

$$\mu(y) \geq \min\ \{\mu(x), \mu(x \rightarrow y)\} = \mu(1),$$

whence $\mu(y) = \mu(1)$. This means that $y \in A_\mu$.

Using a given fuzzy deductive system, we construct a new fuzzy deductive system.

Let $\alpha \geq 0$ be a real number. If $m \in [0, 1], m^\alpha$ shall mean the positive root in case $\alpha < 1$. We define $\mu^\alpha : A \rightarrow [0, 1]$ by $\mu^\alpha(x) = (\mu(x))^\alpha$.

Theorem 3.14 — If μ is a fuzzy deductive system of a Hilbert algebra A, then μ^α is also a fuzzy deductive system of A and $A_{\mu^\alpha} = A_\mu$.

PROOF : We have that $\mu^\alpha(1) = (\mu(1))^\alpha \geq (\mu(x))^\alpha = \mu^\alpha(x)$ for all $x \in A$. Let $x, y \in A$. We assert that $\mu^\alpha(y) \geq \min\ \{\mu^\alpha(x), \mu^\alpha(x \rightarrow y)\}$. In fact, suppose that $\mu(x) \leq \mu(x \rightarrow y)$. It follows from Definition 3.1(ii) that $\mu(y) \geq \mu(x)$. Hence $\mu^\alpha(x) \leq \mu^\alpha(x \rightarrow y)$ and $\mu^\alpha(x) \leq \mu^\alpha(y)$, which imply that $\mu^\alpha(y) \geq \min\ \{\mu^\alpha(x), \mu^\alpha(x \rightarrow y)\}$. The argument is similar if $\mu(x) \geq \mu(x \rightarrow y)$. Finally

$$\begin{aligned} A_{\mu^\alpha} &= \{x \in A \mid \mu^\alpha(x) = \mu^\alpha(1)\} \\ &= \{x \in A \mid (\mu(x))^\alpha = (\mu(1))^\alpha\} \\ &= \{x \in A \mid \mu(x) = \mu(1)\} \\ &= A_\mu. \end{aligned}$$

4. CARTESIAN PRODUCT OF FUZZY DEDUCTIVE SYSTEMS

Let A and B be Hilbert algebras and let

$$A \times B = \{(x, y) \mid x \in A,\ y \in B\}.$$

We define an operation $\rightarrow$ on $A \times B$ by

$$(x, y) \rightarrow (x', y') = (x \rightarrow x', y \rightarrow y') \text{ for all } (x, y), (x', y') \in A \times B.$$

Then we can easily verify that $(A \times B, \rightarrow (1, 1))$ is a Hilbert algebra.

Proposition 4.1 — Let D_1 and D_2 be deductive systems of Hilbert algebras A and B respectively. Then $D_1 \times D_2$ is a deductive system of $A \times B$.

PROOF : Obvious from definition.

Proposition 4.2 — For a given fuzzy set ν in a Hilbert algebra A, let μ_ν be the strongest fuzzy relation on A. If μ_ν is a fuzzy deductive system of $A \times A$, then $\nu(x) \leq \nu(1)$ for all $x \in A$.

PROOF : Since μ_ν is a fuzzy deductive system of $A \times A$, therefore

$$\mu_\nu(x, y) \leq \mu_\nu(1, 1) \text{ for all } (x, y) \in A \times A.$$

But this means that $\min\ \{\nu(x), \nu(y)\} \leq \min\{\nu(1), \nu(1)\}$, which implies that $\nu(x) \leq \nu(1)$ for all $x \in A$.

The following proposition is an immediate consequence of Lemma 2.10, and we omit the proof.

Proposition 4.3 — If ν is a fuzzy deductive system of a Hilbert algebra A, then the level deductive systems of μ_ν are given by $(\mu_\nu)_\alpha = \nu_\alpha \times \nu_\alpha$ for all $\alpha \in [0, 1]$.

Theorem 4.4 — Let μ and ν be fuzzy deductive systems of a Hilbert algebra A. Then $\mu \times \nu$ is a fuzzy deductive system of $A \times A$.

PROOF : First we have that for every $(x, y) \in A \times A$,

$$(\mu \times \nu)(1, 1) = \min \{\mu(1), \nu(1)\} \geq \min \{\mu(x), \nu(y)\} = (\mu \times \nu)(x, y).$$

Now let $(x, y), (x', y') \in A \times A$. Then

$$\begin{aligned}
&\min \{(\mu \times \nu)(x, y),\ (\mu \times \nu)((x, y) \to (x', y'))\} \\
&= \min \{(\mu \times \nu)(x, y),\ (\mu \times \nu)(x \to x', y \to y')\} \\
&= \min \{\min\{\mu(x), \nu(y)\},\ \min \{\mu(x \to x'), \nu(y \to y')\}\} \\
&= \min \{\min \{\mu(x), \mu(x \to x')\},\ \min \{\nu(y), \nu(y \to y')\}\} \\
&\leq \min \{\mu(x'), \nu(y')\} \\
&= (\mu \times \nu)(x', y').
\end{aligned}$$

This completes the proof.

Theorem 4.5 — Let μ and ν be fuzzy sets in a Hilbert algebra A such that $\mu \times \nu$ is a fuzzy deductive system of $A \times A$. Then

(i) either μ or ν satisfies Definition 3.1(i).

(ii) if μ satisfies Definition 3.1(i), then either $\mu(x) \leq \nu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in A$.

(iii) if ν satisfies Definition 3.1(i), then either $\mu(x) \leq \mu(1)$ or $\nu(x) \leq \mu(1)$ for all $x \in A$.

(iv) either μ or ν is a fuzzy deductive system of A.

PROOF : (i) If both μ and ν do not satisfy Definition 3.1(i), then there exist x, $y \in A$ such that $\mu(x) > \mu(1)$ and $\nu(y) > \nu(1)$. Then

$$(\mu \times \nu)(x, y) = \min \{\mu(x), \nu(y)\} > \min \{\mu(1), \nu(1)\} = (\mu \times \nu)(1, 1).$$

This contradicts the fact that $\mu \times \nu$ is a fuzzy deductive system of $A \times A$. Hence (i) holds.

(ii) Assume that μ satisfies Definition 3.1(i) and let x, $y \in A$ be such that $\mu(x) > \nu(1)$ and $\nu(y) > \nu(1)$. Then

$$(\mu \times \nu)(1, 1) = \min \{\mu(1), \nu(1)\} = \nu(1).$$

It follows that $(\mu \times \nu)(x, y) = \min \{\mu(x), \nu(y)\} > \nu(1) = (\mu \times \nu)(1, 1)$, which is a contradiction. Thus (ii) is true.

(iii) This is by similar method to part (ii).

(iv) Since, by (i), either μ or ν satisfies Definition 3.1(i), without loss of generality we may assume that μ satisfies Definition 3.1(i). Using (ii) we have that either $\mu(x) \leq \nu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in A$.

If $\mu(x) \leq \nu(1)$ for all $x \in A$, then

$$(\mu \times \nu)(x, 1) = \min\{\mu(x), \nu(1)\} = \mu(x) \text{ for all } x \in A.$$

Let $(x, y), (x', y') \in A \times A$. Since $\mu \times \nu$ is a fuzzy deductive system of $A \times A$, by Definition 3.1(ii) we have

$$\begin{aligned}(\mu \times \nu)(x', y') &\geq \min\{(\mu \times \nu)(x, y), (\mu \times \nu)((x, y) \to (x', y'))\} \qquad \ldots (*)\\ &= \min\{(\mu \times \nu)(x, y), (\mu \times \nu)(x \to x', y \to y')\}.\end{aligned}$$

If we take $y = y' = 1$, then

$$\begin{aligned}\mu(x') = (\mu \times \nu)(x', 1) &\geq \min\{(\mu \times \nu)(x, 1), (\mu \times \nu)(x \to x', 1 \to 1)\}\\ &= \min\{(\mu \times \nu)(x, 1), (\mu \times \nu)(x \to x', 1)\}\\ &= \min\{\min\{\mu(x), \nu(1)\}, \min\{\mu(x \to x'), \nu(1)\}\}\\ &= \min\{\mu(x), \mu(x \to x')\},\end{aligned}$$

showing that μ satisfies Definition 3.1(ii). Hence μ is a fuzzy deductive system of A.

Now we consider the case $\nu(x) \leq \nu(1)$ for all $x \in A$. Suppose that $\mu(y) > \nu(1)$ for some $y \in A$. Then $\mu(1) \geq \mu(y) > \nu(1)$. Since $\nu(x) \leq \nu(1)$ for all $x \in A$, it follows that $\mu(1) > \nu(x)$ for all $x \in A$. Hence $(\mu \times \nu)(1, x) = \min\{\mu(1), \nu(x)\} = \nu(x)$ for all $x \in A$.

Taking $x = x' = 1$ in (*); then

$$\begin{aligned}\nu(y') = (\mu \times \nu)(1, y') &\geq \min\{(\mu \times \nu)(1, y), (\mu \times \nu)(1 \to 1, y \to y')\}\\ &= \min\{(\mu \times \nu)(1, y), (\mu \times \nu)(1, y \to y')\}\\ &= \min\{\min\{\mu(1), \nu(y)\}, \min\{\mu(1), \nu(y \to y')\}\}\\ &= \min\{\nu(y), \nu(y \to y')\},\end{aligned}$$

which proves that ν satisfies Definition 3.1(ii). Hence ν is a fuzzy deductive system of A. This completes the proof.

Now we give an example to show that if $\mu \times \nu$ is a fuzzy deductive system of $A \times A$, then μ and ν both need not be fuzzy deductive systems of A.

Example 4.6 — Let A be a Hilbert algebra with $|A| \geq 2$ and let $\alpha, \beta \in [0, 1]$ be such that $0 \leq \alpha \leq \beta < 1$. Define fuzzy sets μ and $\nu : A \to [0, 1]$ by $\mu(x) = \alpha$ and

$$\nu(x) = \begin{cases} \beta, & \text{if } x = 1, \\ 1, & \text{if } x \neq 1, \end{cases}$$

for all $x \in A$, respectively. Then $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} = \alpha$ for all $(x, y) \in A \times A$, that is, $\mu \times \nu : A \times A \to [0, 1]$ is a constant function. Hence $\mu \times \nu$ is a fuzzy deductive system of $A \times A$. Now μ is a fuzzy deductive system of A, but ν is not a fuzzy deductive system of A because ν does not satisfy Definition 3.1(i).

Theorem 4.7 — Let ν be a fuzzy set in a Hilbert algebra A and let μ_ν be the strongest fuzzy relation on A. Then ν is a fuzzy deductive system of A if and only if μ_ν is a fuzzy deductive system of $A \times A$.

PROOF : Assume that ν is a fuzzy deductive system of A. We note from Definition 3.1(i) that for all $(x, y) \in A \times A$,

$$\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\} \leq \min\{\nu(1), \nu(1)\} = \mu_\nu(1, 1),$$

showing that μ_ν satisfies Definition 3.1(i). Let $(x, y), (x', y') \in A \times A$. Then

$$\begin{aligned} &\min\{\mu_\nu(x, y), \mu_\nu((x, y) \to (x', y'))\} \\ &= \min\{\mu_\nu(x, y), \mu_\nu(x \to x', y \to y')\} \\ &= \min\{\min\{\nu(x), \nu(y)\}, \min\{\nu(x \to x'), \nu(y \to y')\}\} \\ &= \min\{\min\{\nu(x), \nu(x \to x')\}, \min\{\nu(y), \nu(y \to y')\}\} \\ &\leq \min\{\nu(x'), \nu(y')\} \\ &= \mu_\nu(x', y'), \end{aligned}$$

which proves that μ_ν satisfies Definition 3.1(ii). Hence μ_ν is a fuzzy deductive system of $A \times A$.

Conversely suppose that μ_ν is a fuzzy deductive system of $A \times A$. Then

$$\min\{\nu(x), \nu(y)\} = \mu_\nu(x, y) \leq \mu_\nu(1, 1) = \min\{\nu(1), \nu(1)\} = \nu(1)$$

for all $x, y \in A$. It follows that $\nu(x) \leq \nu(1)$ for all $x \in A$. For any $(x, y), (x', y') \in A \times A$, we have that

$$\begin{aligned} \min\{\nu(x'), \nu(y')\} &= \mu_\nu(x', y') \\ &\geq \min\{\mu_\nu(x, y), \mu_\nu((x, y) \to (x', y'))\} \\ &= \min\{\mu_\nu(x, y), \mu_\nu(x \to x', y \to y')\} \\ &= \min\{\min\{\nu(x), \nu(y)\}, \min\{\nu(x \to x'), \nu(y \to y')\}\} \\ &= \min\{\min\{\nu(x), \nu(x \to x')\}, \min\{\nu(y), \nu(y \to y')\}\}. \end{aligned}$$

In particular, if we take $y = y' = 1$ (resp. $x = x' = 1$) then

$$\nu(x') \geq \min\{\nu(x), \nu(x \to x')\}$$

$$(\text{resp. } \nu(y') \geq \min\{\nu(y), \nu(y \to y')\}).$$

The proof is complete.

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