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ON FOURIER-JACOBI TRANSFORMATION

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Mizony[4] has studied Fourier-Jacobi transformation. In this paper we have constructed test spaces $Z^a_{\alpha,\beta}\,\Psi_{a,\alpha,\beta}$ and the dual spaces. We define the Fourier-Jacobi transformation on the dual space using Parseval relation following the technique of Gel'fand and Shilov[3] and study some operation transform formulae related to the Laplace differential operator $\Delta_{\alpha,\beta}$ and its adjoint $\Delta_{\alpha,\beta}$.

1. INTRODUCTION

We recall some results from Mizony[4].

For $f \in L^1(\mathbb{R}_+, W_{\alpha,\beta}(t)\,dt)$, $\mathrm{Re}(\alpha) > -1$, $\lambda \in \mathbb{R}_+$ then the Fourier-Jacobi transformation of f is given by

$$\mathcal{F}_{\alpha,\beta}[f](\lambda) =$$

$$F(\lambda) = 2^{2\rho+0.5}[\Gamma(\alpha+1)]^{-1}\int_0^{\infty} f(t)\,\phi_{\alpha,\beta}(\lambda,t)\,W_{\alpha,\beta}(t)\,dt \qquad \ldots (1.1)$$

where

$$w = W_{\alpha,\beta}(t) = \sinh^{(2\alpha+1)}(t)\cdot\cosh^{(2\beta+1)}(t) \qquad \ldots (1.2)$$

is a measure on $\mathbb{R}_+$ related to a pair (α, β) of complex numbers.
For $\alpha \neq -1, -2, -3, \ldots, \alpha, \beta, \lambda \in \mathbb{C}$, $i\lambda \notin \mathbb{Z}$ and $t \in \mathbb{R}_+$

$$\phi_{\alpha,\beta}(\lambda,t) = {}_2F_1((\rho+i\lambda)/2,\ (\rho-i\lambda)/2;\ \alpha+1;\ -\sinh^2(t)) \qquad \ldots (1.3)$$

is the Jacobi function of first kind which is one of the solutions of the differential operator equation

$$(\Delta_{\alpha,\beta} + \lambda^2 + \rho^2) f = 0 \qquad \ldots (1.4)$$

where the Laplacian operator

$$\Delta_{\alpha,\beta} = w^{-1} DwD = D^2 + (w'/w)D \qquad \text{... (1.5)}$$

with $D \equiv d/dt$ and $\rho = \alpha + \beta + 1$. The partic of (1.1) are :

(i) For $\alpha = \beta = -0.5$, (1.1) is a Fo transform.

(ii) When $\alpha = (n-2)/2$, $n \in N^*$, $\beta = -0.5$, (1.1) gives a Fourier-Jacobi Spherical transformation associated with the group $SO_0(n, 1)$.

(iii) For $\alpha = n-1$, $n \in N^*$, $\beta = 0$ then (1.1) reduces to the Fourier Spherical transformation related to the group $SU(n, 1)$.

(iv) If $\alpha = 2n - 1$ and $\beta = 1$, we get (1.1) in the form of Fourier Spherical transformation associated with $SP(n, 1)$, $n = 2, 3, 4, \ldots$

(v) For $\alpha = 7$, $\beta = 3$, (1.1) becomes a transformation associated with exceptional group $F4(-20)$.

The classical inversion formula for the Fourier-Jacobi transform[4] is as below :

When Re $(\alpha) > -0.5$, $|\operatorname{Re}(\beta)| < \operatorname{Re}(\alpha + 1)$ and $f \in L^1(\mathbb{R}_+, [C_{\alpha,\beta}(\lambda) C_{\alpha,\beta}(-\lambda)]^{-1} d\lambda)$, then

$$\mathcal{F}^{-1}_{\alpha,\beta}[F](t) = f(t) = \sqrt{2}\,[\Gamma(\alpha+1)]^{-1} \times \int_0^\infty f(\lambda)\,\phi_{\alpha,\beta}(\lambda, t)/[C_{\alpha,\beta}(\lambda)\, C_{\alpha,\beta}(-\lambda)]\, d\lambda \qquad \text{... (1.6)}$$

where

$$C_{\alpha,\beta}(\lambda) = 2^\rho\, \Gamma(i\lambda/2)\, \Gamma[(1+i\lambda)/2] \times \{\Gamma[(\alpha+\beta+1+i\lambda)/2]\, \Gamma[\alpha-\beta+1+i\lambda)/2]\}^{-1} \qquad \text{... (1.7)}$$

is Harishchandra function. The Bessel Parseval formula is as below :

When $\alpha, \beta \in \mathbb{R}$, $|\beta| < \alpha + 1$ then the Fourier-Jacobi transformation is an isometric isomorphism from $L^2(\mathbb{R}_+, 2^{2\rho} W_{\alpha,\beta}(t)\, dt)$ on to $L^2(\mathbb{R}_+, |C_{\alpha,\beta}(\lambda)|^{-2}\, d\lambda)$ and for all $f, g \in L^2(\mathbb{R}_+, 2^{2\rho} W_{\alpha,\beta}(t)\, dt)$

$$\int_0^\infty f(t)\, g(t)\, 2^{2\rho}\, W_{\alpha,\beta}(t)\, dt = \int_0^\infty F(\lambda)\, G(\lambda)\, |C_{\alpha,\beta}(\lambda)|^{-2}\, d\lambda. \qquad \text{... (1.8)}$$

Observe that for $\operatorname{Re}(\alpha) > -1/2$

$W_{\alpha,\beta}(t) = o(1)$ as $t \to 0+$ and $W_{\alpha,\beta}(t) = O(e^{2\rho t})$ as $t \to \infty$ so that

$$W_{\alpha,\beta}(t) = o(e^{(2\rho+1)t}) \text{ as } t \to \infty. \qquad \text{... (1.9)}$$

The differential operator $\Delta^k_{\alpha,\beta}$ for any $k = 0, 1, 2, \ldots$ can be written in the form

$$\Delta^k_{\alpha,\beta}(\phi) = \sum_{j=1}^{2k} b_j(t)\, d^j\phi/dt^j \qquad \text{... (1.10)}$$

where $b_{2k} = 1$ and remaining $b_j(t)$ are functions of w'/w and its derivatives. Also $w'/w = O(1)$ as $t \to \infty$. ... (1.11)

The result (9) in Erdelyi *et al.*[2] (p.76) says for the large values of t the function $\phi_{\alpha,\beta}(\lambda, t)$ can be put in the form

$$\lambda_1 \sinh^{-\rho - i\lambda}(t) + \lambda_2 \sinh^{-\rho + i\lambda} + O(\sinh t)^{-\rho - i\lambda - 2} + O(\sinh t)^{-\rho + i\lambda - 2}$$

... (1.12)

where λ_1, λ_2 are constants and $i\lambda$ is not an integer. For small values of t, $W_{\alpha,\beta}(t)$ and $\phi_{\alpha,\beta}(\lambda, t)$ take values close to 0 and 1 respectively.

2. The Spaces $Z^a_{\alpha,\beta}$ and $Z'^a_{\alpha,\beta}$

Let $I = (0, \infty)$ and $-\rho < a < \infty$. $Z^a_{\alpha,\beta} \equiv Z_a$ be a space consisting of complex-valued infinitely differentiable functions $\theta(t)$ on I such that the functionals

$$\eta^a_k(\theta) = \eta^a_{\alpha,\beta,k}(\theta) = \sup_{t \in I} \left| e^{at} \Delta^k_{\alpha,\beta} W_{\alpha,\beta}(t)\, \theta(t) \right| \qquad \text{... (2.1)}$$

for each $k = 0, 1, 2, \ldots$, assume finite values. The topology of Z_a is determined by $\left\{ \eta^a_k \right\}^{\infty}_{k=0}$.

Z_a is a countably multinormed space and is complete and hence a Frechet space. Z_a is a testing function space.

The collection Z_a' of continuous linear functionals on Z_a is defined as dual space of Z_a. With the usual addition and multiplication by a complex number Z_a' is a linear space.

Remark : (I) $\mathcal{D}(I) \subset Z_a$. Convergence in $\mathcal{D}(I)$ implies the convergence in Z_a. Restriction of any member of Z_a' to $\mathcal{D}(I)$ is a member of $\mathcal{D}'(I)$. Hence Z_a' is the space of distributions and is complete due to the completeness of $Z_{a'}$.

(II) $Z_a \subset \mathcal{E}(I)$. Moreover Z_a is dense in $\mathcal{E}(I)$ because $\mathcal{D}(I) \subset Z_a \subset \mathcal{E}(I)$ and $\mathcal{D}(I)$ is dense in $\mathcal{E}(I)$. Topology of Z_a is stronger than the topology induced on it by $\mathcal{E}(I)$. Then every generalized function on I with compact support is in Z_a'.

(III) When $f(t)$ is locally integrable function such that $f(t)\,[w(t)\, e^{at}]^{-1}$ is absolutely integrable on I, then $f(t)$ generates a regular member of Z_a' through the definition

$$\langle f, \theta \rangle = \int_0^{\infty} f(t)\, \theta(t)\, dt \qquad \theta \in Z_a.$$

3. The Space $\Psi_{a,\alpha,\beta}$ of Fourier-Jacobi Transforms

Proposition 3.1 — If $-\rho < a < \infty$, α, β real and if $\theta \in Z_a$, then $\mathcal{F}_{\alpha,\beta}(\theta)$ defined by (1.1) exists.

PROOF : For every $\theta \in Z_a$ by (1.1) we have

$$\mathcal{F}_{\alpha,\beta}(\theta) = 2^{2\rho+0.5}[\Gamma(\alpha+1)]^{-1} \int_0^\infty \theta(t)\, \phi_{\alpha,\beta}(\lambda, t)\, W_{\alpha,\beta}(t)\, dt.$$

$$|\mathcal{F}_{\alpha,\beta}(\theta)| \leq 2^{2\rho+0.5}[\Gamma(\alpha+1)]^{-1} \int_0^\infty |e^{at} w(t)\, \theta(t)|\; |\phi_{\alpha,\beta}(\lambda, t)\, e^{-at}|\, dt$$

$$\leq 2^{2\rho+0.5}\, \eta_0^a(\theta)\, [\Gamma(\alpha+1)]^{-1} \int_0^\infty |\phi_{\alpha,\beta}(\lambda, t)\, e^{-at}|\, dt.$$

From the order relation (1.12) and since $-\rho < a < \infty$, the integrand $|\phi_{\alpha,\beta}(\lambda, t)\, e^{-at}|$ is bounded and so the integral on the right-hand side is finite. Hence $\mathcal{F}_{\alpha,\beta}(\theta)$ exists.

Definition 3.2 — For $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\lambda \in \mathbb{R}_+$, we construct a space $\Psi_{a,\alpha,\beta} = \Psi_a$ as below

$$\Psi_a = \{\mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)|^{-2} / \theta \in Z_a\} \qquad \dots (3.1)$$

where $C_{\alpha,\beta}(\lambda)$ is the Harischandra function defined in (1.7). Ψ_a is a linear space under the usual addition and a multiplication by a complex number.

We define a seminorm on Ψ_a as below :

If $\phi \in \Psi_a$ then

$$\gamma_k^a(\phi) = \gamma_{\alpha,\beta,k}^a(\phi) = \gamma_{\alpha,\beta,k}^a(\mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)|^{-2}) = \eta_k^a(\theta) \qquad \dots (3.2)$$

for some $\theta \in Z_a$ satisfying $\phi(\lambda) = \mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)|^{-2}$.

Remark 3.3 : γ_0^a is a norm on Ψ_a; so that $\{\gamma_k^a\}_{k=0}^\infty$ is a countable multinorm on Ψ_a. The countably multinormed space Ψ_a is also complete and hence a Frechet space. Topology of Ψ_a is determined by the multinorm $\{\gamma_k^a\}_{k=0}^\infty$.

Definition 3.4 — We define the space Z_{*a} as follows :

$$w(t)\, \theta(t) \in Z_{*a} \text{ if and only if } \theta \in Z_a. \qquad \dots (3.3)$$

It is clear that Z_{*a} is a linear space under usual addition and multiplication by a complex number. We define a seminorm η_k^{*a} on Z_{*a} by .

$$\eta_k^{*a}(\xi) = \eta_k^a(w^{-1}\xi) = \eta_k^a(\theta) \qquad k = 0, 1, 2, 3, \ldots \qquad \ldots (3.4)$$

where $\xi = w(t)\,\theta(t) \in Z_{*a}$ and $\theta \in Z_a$.

η_0^{*a} is a norm hence the sequence $\{\eta_k^{*a}\}_{k=0}^{\infty}$ is a countable multinorm on Z_{*a}. Z_{*a} is complete and hence a Frechet space.

PROOF : Z_{*a} is a countable multinormed space. For the completeness consider a Cauchy sequence $\{\xi_\nu\}_{\nu=1}^{\infty}$ in Z_{*a}. Then $\{w^{-1}\xi_\nu\}_{\nu=1}^{\infty} \equiv \{\theta_\nu\}_{\nu=1}^{\infty}$ will be a Cauchy sequence in Z_a which is convergent in it since Z_a is a Frechet space. Using the Definition 3.4 we conclude that $\{\xi_\nu\}_{\nu=1}^{\infty}$ is convergent in Z_{*a}. Q.E.D.

Z_{*a} is a testing function space.

Definition 3.5 — The collection of all linear and continuous functionals on Z_{*a} is defined as the dual space of Z_{*a} and is denoted by Z'_{*a}.

The topology of Z_{*a} is generated by the sequence of multinorm $\{\eta_k^{*a}\}_{k=0}^{\infty}$. Since Z_{*a} is a testing function space Z'_{*a} is the space of generalized functions and is complete due to the completeness of Z_{*a}.

For any $\xi(t) = w(t)\,\theta(t) \in Z_{*a}$ with some $\theta(t) \in Z_a$, and for any nonnegative integer r, we set

$$\mu_r^*(\xi) = \max_{0 \le k \le r} \eta_k^*(\xi) = \max_{0 \le k \le r} \eta_k(w^{-1}\xi) = \max_{0 \le k \le r} \eta_k(\theta) = \mu_r(\theta).$$

Then for each $f \in Z'_{*a}$ there exist a constant C and a nonnegative integer r such that

$$|\langle f(t), w(t)\,\theta(t)\rangle| \le C\,\mu_r(\theta).$$

If $f(t)$ is locally and absolutely integrable on I then $f(t)$ generates a regular generalized function in Z'_{*a} by the formula

$$\langle f(t), w(t)\,\theta(t)\rangle = \int_0^{\infty} f(t)\,w(t)\,\theta(t)\,dt, \qquad \theta \in Z_a.$$

Proposition 3.6 — The mapping $S : \Psi_a \to Z_{*a}$ defined by

$$S\left(\mathcal{F}_{\alpha,\beta}(\theta)\,|\,C_{\alpha,\beta}(\lambda)|^{-2}\right) = 2^{2\rho}\,\theta(t)\,w(t)$$

is an isomorphism.

PROOF : Linearity is obvious. For continuity consider a sequence $\{\phi_\nu\}_1^{\infty}$ in Ψ_a which converges to zero. Then for any $k = 0, 1, 2, \ldots$ and $-\rho < a < \infty$,

$\gamma_k^a(\phi_\nu) \to 0$ as $\nu \to \infty$ uniformly on I. That is to say $\gamma_k^a\left(\mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)\mid^{-2}\right) \to 0$ as $\nu \to \infty$ uniformly on I for some sequence $\{\theta_\nu\}$ of functions in Z_a. With the help of Definition (3.2) of seminorm this is the same thing as to say that $\eta_k^a(\theta_\nu) \to 0$ as $\nu \to \infty$ uniformly on I for any $k = 0, 1, 2, 3, \ldots$ and $-\rho < a < \infty$. Thus $\eta_k^{*a}(w\theta_n) \to 0$ as $n \to \infty$. The continuity is proved by the Lemma 1.10.2 of Zemanian[5] (p.27). Also S is one-one and onto. In this way we proved that S is continuous, linear, one-one mapping from Ψ_a on to Z_{*a}. In similar way we can prove that the mapping $S^{-1} : Z_{*a} \to \Psi_a$ defined by

$$2^{2\rho}\, w(t)\, \theta(t) \to \mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)\mid^{-2} = \phi(\lambda) \qquad \ldots (3.5)$$

for $\theta \in Z_a$ and corresponding $\phi \in \Psi_a$, is linear and continuous. For continuity consider a sequence $\left\{\theta_\nu\right\}_1^\infty$ in Z_a which converges to zero. Then for any $k = 0, 1, 2, 3, \ldots$ and $-\rho < a < \infty$, $\eta_k^a(\theta_\nu) \to 0$ as $\nu \to \infty$ uniformly on I. That is $\gamma_k^a(\mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)\mid^{-2}) \to 0$ as $\nu \to \infty$ uniformly on I. Same thing is to say that $\gamma_k^a(\theta_\nu) \to 0$ as $\nu \to \infty$ uniformly on I, for the corresponding sequence $\{\phi_\nu\} = \{\mathcal{F}_{\alpha,\beta}(\theta_\nu) \mid C_{\alpha,\beta}(\lambda).\mid^{-2}\}$. This leads us to the conclusion that $\left\{\phi_\nu\right\}_1^\infty$ converges to zero uniformly on I. Hence $S : \Psi_a \to Z_{*a}$ is an isomorphism.

4. Generalized Fourier-Jacobi Transformation

Definition 4.1 — For $\alpha, \beta \in \mathbb{R}$, $\alpha > -1$, $\lambda \in \mathbb{R}_+$, $\theta \in Z_a$, we define a Fourier-Jacobi transform $\mathcal{F}'_{\alpha,\beta}[f]$ of a generalized function $f \in Z'_{*a}$ by a formula

$$\langle \mathcal{F}'_{\alpha,\beta}(f),\ \mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)\mid^{-2} \rangle = \langle f, \theta(t)\, 2^{2\rho}\, w(t) \rangle. \qquad \ldots (4.1)$$

Since $\theta \in Z_a$, $2^{2\rho}\, w(t)\, \theta(t) \in Z_{*a}$. Therefore since $f \in Z'_{*a}$ the right-hand side of (4.1) has sense. Again since the mapping $\mathcal{F}_{\alpha,\beta}(\theta) \mid C_{\alpha,\beta}(\lambda)\mid^{-2} \to 2^{2\rho}\, w(t)\, \theta(t)$ is an isomorphism from ψ_a to Z_{*a}, then the mapping of $f \to \mathcal{F}'_{\alpha,\beta(f)}$ from Z'_{*a} to ψ'_a is an isomorphism. Our definition of Fourier-Jacobi transform is consistent with the Parseval equation.

Let $\mathcal{F}'_{\alpha,\beta}[f] = g(\lambda)$ and $\mathcal{F}_{\alpha,\beta}[\theta] = \phi(\lambda)$ then $f(t) = \mathcal{F}'^{-1}_{\alpha,\beta}[g]$ and $\phi = \mathcal{F}^{-1}_{\alpha,\beta}[\theta]$. In view of this (4.1) becomes

$$\langle \mathcal{F}'^{-1}_{\alpha,\beta}[g],\ \mathcal{F}^{-1}_{\alpha,\beta}[\theta]\, 2^{2\rho}\, w(t) \rangle = \langle g(\lambda),\ \phi(\lambda) \mid C_{\alpha,\beta}(\lambda)\mid^{-2} \rangle. \qquad \ldots (4.2)$$

Lemma 4.2 — The mapping T defined by $\theta \to (w^{-1}\, \Delta_{\alpha,\beta}\, w)\theta$ is an automorphism of Z_a, where $w = W_{\alpha,\beta}(t)$, $0 < t < \infty$.

Proof : Let $\theta \in Z_a$. Then $\theta(t)$ is a complex-valued, infinitely differentiable function, so is $w^{-1}\Delta_{\alpha,\beta}w\theta(t)$. Now

$$\eta_k^a(w^{-1}\Delta_{\alpha,\beta}w\theta) = \sup_{t\in I}|e^{at}\Delta_{\alpha,\beta}^k w(w^{-1}\Delta_{\alpha,\beta}w)\theta(t)|$$

$$= \sup_{t\in K}|e^{at}\Delta_{\alpha,\beta}^{k+1}w\theta(t)| = \eta_{k+1}^a(\theta) \quad \text{... (4.3)}$$

which is finite. Thus $(w^{-1}\Delta_{\alpha,\beta}w)\theta\in Z_a$ whenever $\theta\in Z_a$. The relation (4.3) establishes that a linear mapping T is continuous on Z_a. Linearity of T implies if $T(\theta_1)=T(\theta_2)$ then $T(\theta_1-\theta_2)=0$. That is $(w^{-1}\Delta_{\alpha,\beta}w)(\theta_1-\theta_2)=0$. Which ultimately gives $\theta_1=\theta_2$.

This proves T is injective also T is onto.

In similar way we can prove that the inverse mapping T^{-1} is continuous and linear. Hence T is an isomorphism from Z_a on to itself. Q.E.D.

Lemma 4.3 — $\phi_{\alpha,\beta}(\lambda,t)/W_{\alpha,\beta}(t)$ is a member of Z_a when $a<\rho$.

PROOF : Obviously $\phi_{\alpha,\beta}(\lambda,t)/w$ is infinitely differentiable complex-valued function. For a nonnegative integer k and $a<\rho$

$$\eta_k^a(\phi_{\alpha,\beta}(\lambda,t)/w) = \sup_{t\in I}|e^{at}\Delta_{\alpha,\beta}^k w\,\phi_{\alpha,\beta}(\lambda,t)/w|$$

$$= \sup_{t\in I}|e^{at}\Delta_{\alpha,\beta}^k\phi_{\alpha,\beta}(\lambda,t)|$$

Using (1.4) we get

$$= \sup_{t\in I}|e^{at}(-1)^k(\lambda^2+\rho^2)^k\phi_{\alpha,\beta}(\lambda,t)|$$

$$= (\lambda^2+\rho^2)^k\sup_{t\in I}|e^{at}\phi_{\alpha,\beta}(\lambda,t)|.$$

The order relation (1.12) shows that when $a<\rho$, $\sup_{t\in I}|e^{at}\phi_{\alpha,\beta}(\lambda,t)|$ is bounded. So that $\eta_k^a(\phi_{\alpha,\beta}(\lambda,t)/w)$ is finite; showing further that $\phi_{\alpha,\beta}(\lambda,t)/w$ is a member of Z_a.

Theorem 4.4 — If $f\in Z_a$ and $a<\rho$ then for a nonnegative integer k

$$\langle(\Delta'_{\alpha,\beta})^k f,\ \phi_{\alpha,\beta}(\lambda,t)/w\rangle = (-1)^k(\lambda^2+\rho^2)^k\langle f,\phi_{\alpha,\beta}(\lambda,t)/w\rangle \quad \text{... (4.4)}$$

where $\Delta'_{\alpha,\beta}$ is an adjoint operator defined as

$$\langle\Delta'_{\alpha,\beta}f,\phi\rangle = \langle f,(w^{-1}\Delta_{\alpha,\beta}w)\phi\rangle. \quad \text{... (4.5)}$$

PROOF : Due to Lemma 4.3 the right hand side of (4.4) has sense. Also by Lemma 4.2, $w^{-1}\Delta_{\alpha,\beta}w\theta\in Z_a$ whenever $\theta\in Z_a$ which make sense to the right hand side of (4.5). Then

$$\langle (\Delta'_{\alpha,\beta}) f, \phi_{\alpha,\beta}(\lambda, t)/w \rangle = \langle (\Delta'_{\alpha,\beta})^{k-1} f, (w^{-1} \Delta_{\alpha,\beta} w)(\phi_{\alpha,\beta}(\lambda, t)/w) \rangle$$

$$= \langle (\Delta'_{\alpha,\beta})^{k-1} f, w^{-1}(-1)(\lambda^2 + \rho^2) \phi_{\alpha,\beta}(\lambda, t) \rangle$$

$$= (-1)(\lambda^2 + \rho^2) \langle (\Delta'_{\alpha,\beta})^{k-2} f, (w^{-1} \Delta_{\alpha,\beta} w) \phi_{\alpha,\beta}(\lambda, t)/w \rangle$$

$$= (-1)^2 (\lambda^2 + \rho^2)^2 \langle (\Delta'_{\alpha,\beta})^{k-2} f, w^{-1} \phi_{\alpha,\beta}(\lambda, t) \rangle.$$

Proceeding in this way we come to the conclusion (4.4).

Remark 4.5 : The special cases of the results in this paper can be obtained by parameterizing α, β as stated in the introduction part of this paper.

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