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SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION

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Some applications of differential subordination are given for certain classes of functions defined through Ruscheweyh derivatives, and for the classes of starlike and convex functions associated with the parabolic region $\text{Re}\{w\} > | w - 1 |$.

1. INTRODUCTION

For positive integers p, n, let $A_n(p)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k$$

which are analytic in the unit disk $U = \{z; | z | < 1\}$. Denote the class $A_1(p)$ by $A(p)$ and the class $A(1)$ by A. Let $H[a, n]$ denote the class of all functions of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k$$

which are analytic in U where a is a complex number. Let $h(z)$ be a univalent analytic function with $h(0) = 1$. Let f and F be analytic in U. Then f is subordinate to F (written $f \prec F$ or $f(z) \prec F(z)$) if $f(0) = F(0)$ and $f(U) \subseteq F(U)$. Denote the subclasses of $A_n(p)$ consisting of functions $f(z)$ for which $\frac{1}{p}\frac{zf'(z)}{f(z)} \prec h(z)$ and $\frac{1}{p}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \prec h(z)$ by $ST_{n,p}(h)$ and $CV_{n,p}(h)$ respectively. These classes have a natural generalization which are defined through the Ruscheweyh derivatives. Some inclusion results and sufficient conditions are obtained for these classes which either extends or improves the earlier results of Chen and Lan[1], Miller and Mocano[3].

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A function $f(z) \in A$ is uniformly convex of order α with $-1 < \alpha < 1$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ lies in the parabolic region Re $\{w - \alpha\} > |w - 1|$. In other words, the function $f(z)$ is uniformly convex of order α if

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{2(1-\alpha)}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2.$$

This class was introduced by Rϕnning[5] and it is denoted by $UCV(\alpha)$. The class of all analytic functions $f(z) \in A$ for which $\frac{zf'(z)}{f(z)}$ lies in the parabolic region is denoted by $S_p(\alpha)$. These classes are extended to the class of p-valent analytic functions and some sharp inclusion results for these new classes are obtained.

In order to prove our theorems we need the following results of Miller and Mocanu[3].

Theorem A — Let $\alpha > 0$, n a positive integer, and let $q(z)$ be univalent with $q(0) = 1$ and $q(z) \neq 0$. Set

$$Q(z) = \frac{zq'(z)}{q(z)} \text{ and } h(z) = q(z) + \alpha n Q(z)$$

and suppose that

(a) Re $\left[\frac{h'(z)q(z)}{q'(z)}\right] > 0$, $z \in U$ and either

(b) $h(z)$ is convex, or

(b′) $\log q(z)$ is convex ($Q(z)$ is starlike).

If $B \in H[1, n]$ and $B \prec h$, then the analytic solution of

$$\alpha z p'(z) + B(z)p(z) = 1$$

is given by

$$p(z) = \alpha^{-1} z^{-1/\alpha} e^{-b(z)} \int_0^z e^{b(t)} t^{1/\alpha - 1}\, dt$$

where

$$b(z) = \alpha^{-1} \int_0^z [B(t) - 1]\, t^{-1}\, dt,$$

satisfies $p \in H[1, n]$, $p(z) \neq 0$ and $p \prec \frac{1}{q}$. The subordination is sharp.

Theorem B — Let $\alpha > 0$, n a positive integer and let $q(z)$ be analytic in U with $q(0) = 1$, and set

$$h(z) = q(z) + \alpha n \frac{zq'(z)}{q(z)}.$$

If

(i) Re $q(z) > 0$, for $z \in U$ and

(ii) $h(z)$ is convex or $\log q(z)$ is convex,

then

$$\text{Re}\left[\frac{h'(z)\, q(z)}{q'(z)}\right] > 0 \text{ for } z \in U$$

and both $\log q(z)$ and $q(z)$ are univalent.

2. Results

Theorem 2.1 — Let n be a positive integer, and let $q(z)$ be univalent with $q(0) = 1$ and $q(z) \neq 0$. Set

$$h(z) = q(z) + \frac{n}{p}\frac{zq'(z)}{q(z)}$$

and suppose that $q(z)$ and $h(z)$ satisfy the conditions (a) and (b) or (a) and (b′) of Theorem A. If $f \in A_n(p)$, then $f \in CV_{n,p}(h)$ implies $f \in ST_{n,p}(q)$.

PROOF : For the function $f(z) \in CV_{n,p}(h)$, define $B(z)$ by

$$B(z) = \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right),$$

and let $\alpha = 1/p$. Then $\alpha > 0$, $B \in H[1, n]$ and $B(z) \prec h(z)$. Note that the functions $q(z)$ and $h(z)$ satisfy the conditions of Theorem A with $\alpha = 1/p$. Since

$$b(z) = \alpha^{-1} \int_0^z [B(t) - 1]\, t^{-1}\, dt$$

$$= p \int_0^z \left[\frac{1}{p}\left(1 + \frac{tf''(t)}{f'(t)}\right) - 1\right]\frac{dt}{t}$$

$$= \log\left(\frac{z^{1-p} f'(z)}{p}\right),$$

by an application of Theorem A, the analytic solution of the differential equation

$$\alpha z p'(z) + B(z)\, p(z) = 1$$

given by

$$p(z) = \frac{p}{zf'(z)} \int_0^z \frac{f'(t)}{t^{p-1}}\, t^{p-1}\, dt$$

$$= \frac{pf(z)}{zf'(z)}$$

satisfies $\dfrac{1}{p(z)} \prec q(z)$. This is equivalent to $f \in ST_{n,p}(q)$.

Theorem 2.2 — Let $\alpha > 0$, n a positive integer, and let $q(z)$ be univalent with $q(0) = 1$ and $q(z) \neq 0$. Set

$$Q(z) = \frac{zq'(z)}{q(z)} \text{ and } h(z) = q(z) + \frac{\alpha n}{p} Q(z)$$

and suppose that $q(z)$ and $h(z)$ satisfy the conditions (a) and (b) or (a) and (b′) of Theorem A. For $g \in A_n(p)$, define $f(z)$ by

$$f(z) = \left[\frac{p}{\alpha} \int_0^z g(t)^{1/\alpha} t^{-1} \, dt \right]^{\alpha}.$$

Then (i) $f \in A_n(p)$; (ii) $g \in ST_{n,p}(h)$ implies $f \in ST_{n,p}(q)$. The result is sharp.

PROOF OF THEOREM 2.2 : Since $g \in ST_{n,p}(h)$, the function $B(z) = \dfrac{zg'(z)}{pg(z)}$ is well-defined, $B(z) \in H[1, n]$ and $B(z) \prec h(z)$. Since

$$\frac{p}{\alpha} \int_0^z \frac{B(t) - 1}{t} \, dt = \log \left(\frac{g(z)}{z^p} \right)^{1/\alpha},$$

by Theorem A, the differential equation

$$\frac{\alpha}{p} zp'(z) + B(z) p(z) = 1$$

has the analytic solution

$$p(z) = \frac{p}{\alpha g(z)^{1/\alpha}} \int_0^z \frac{g(t)^{1/\alpha}}{t} \, dt$$

and $1/p(z) \prec q(z)$. Since $f(z) = g(z) [p(z)]^{\alpha}$, a computation shows that

$$\frac{1}{p} \frac{zf'(z)}{f(z)} = \frac{1}{p} \frac{zg'(z)}{g(z)} + \frac{\alpha}{p} \frac{zp'(z)}{p(z)}$$

$$= B(z) + \frac{\alpha}{p} \frac{zp'(z)}{p(z)}$$

$$= \frac{1}{p(z)}.$$

This shows that $\dfrac{1}{p} \dfrac{zf'(z)}{f(z)} \prec q(z)$.

In particular we have the following result due to Miller and Mocanu[3].

Corollary 2.1 — Let $q(z)$ be univalent with $q(0) = 1$ and $q(z) \neq 0$. Set

$$h(z) = q(z) + n\frac{zq'(z)}{q(z)}$$

and suppose that $q(z)$ and $h(z)$ satisfy the conditions (a) and (b) or (a) and (b') of Theorem A. If $f \in A_n$, then $f \in CV(h)$ implies $f \in ST(q)$.

A function $f(z) \in A_n(p)$ is p-valent uniformly convex of order β with $-1 < \beta < 1$ if and only if $\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)$ lies in the parabolic region $\mathrm{Re}\, w - \beta > |w - 1|$. It is equivalent to

$$\frac{1}{p}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \prec 1 + \frac{2(1-\alpha)}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2.$$

This class is denoted by $UCV_{n,p}(\beta)$. The class $SP_{n,p}(\beta)$ is defined in a similar manner. It is known that $UCV \subset S_p$. The following result gives the sharp form of the above result for a more general classes of p-valent functions. In view of Theorem 2.1 and Theorem B, the following result is obtained.

Corollary 2.2 — Suppose $f \in UCV_{n,p}(\beta)$ and $q(z)$ be defined by

$$q(z) + n\frac{zq'(z)}{q(z)} = 1 + \frac{2(1-\beta)}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2.$$

Then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

The result is sharp.

For two analytic function $f(z) = z^p + \sum_{k=n+p}^{\infty} \alpha_k z^k$ and $g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k$, their Hadamard product or Convolution $(f * g)(z)$ is defined by $(f * g)(z) = z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k$. For $\delta \geq -p$, the Ruscheweyh derivative of order $\delta + p - 1$ is given by

$$D^{\delta+p-1} f(z) = \frac{z^p}{(1-z)^{\delta+p}} * f(z).$$

The following theorem, a natural extension of the above results to the class of functions defined through the Ruscheweyh derivatives, improves an earlier result of Chen and Lan[1].

Theorem 2.3 — Let δ, p, α be real numbers such that $\delta + p > 0$, $\delta + p + 1 > \alpha > 0$ and n be a positive integer. Suppose $p(z)$ be univalent with $p(0) = 1$ and $p(z) \neq 0$. Set

$$h(z) = \frac{\alpha}{\delta+p+1} + \frac{\delta+p+1-\alpha}{\delta+p+1}p(z) + \frac{\alpha n}{\delta+p+1}\frac{zp'(z)}{p(z)},$$

and suppose that $q(z)$ and $h(z)$ satisfy the conditions (a) and (b) or (a) and (b') of Theorem A. If $f \in A_n(p)$, then

$$\alpha \frac{D^{\delta+p+1} f(z)}{D^{\delta+p} f(z)} + (1-\alpha) \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \prec h(z)$$

implies

$$\frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \prec q(z).$$

Theorem 2.4 — Suppose $\delta, p, \alpha, n, q(z), h(z)$ satisfy the conditions of Theorem 2.3, and let

$$k(z) = z \exp \int_0^z (q(t)-1)\, t^{-1}\, dt.$$

If $f \in A_n(p)$ satisfies

$$\alpha \frac{D^{\delta+p+1} f(z)}{D^{\delta+p} f(z)} + (1-\alpha) \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \prec h(z)$$

where

$$h(z) = \frac{\alpha(1+n)}{\delta+p+1} + \left[1 - \frac{\alpha(1+n)}{\delta+p+1}\right] \frac{zk'(z)}{k(z)} + \frac{\alpha n}{\delta+p+1} \frac{zk''(z)}{k'(z)}$$

then

$$\frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \prec \frac{zk'(z)}{k(z)}.$$

Proof of Theorem 2.3 — Define $h_1(z)$ by

$$h_1(z) = \frac{(\delta+p+1)\, h(z) - \alpha}{\delta+p+1-\alpha}.$$

Then $h_1(z)$ satisfies the conditions

$$h_1(z) = p(z) + \frac{n\alpha}{\delta+p+1-\alpha} \frac{zp'(z)}{p(z)}.$$

Since $\delta + p + 1 > \alpha > 0$, conditions (a) and (b) of Theorem A are satisfied by $h_1(z)$ whenever $h(z)$ satisfies the conditions.

Let $B(z)$ be defined by

$$B(z) = \frac{(\delta+p+1)}{\delta+p+1-\alpha} \left[\alpha \frac{D^{\delta+p+1} f(z)}{D^{\delta+p} f(z)} + (1-\alpha) \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)}\right] - \frac{\alpha}{\delta+p+1}.$$

Then the function $B(z)$ is in $H[1, n]$ and $B(z) \prec h_1(z)$. The differential equation

$$B(z)\,p(z) + \frac{\alpha}{\delta + p + 1 - \alpha}\, zp'(z) = 1$$

has the solution given by

$$p(z) = \frac{D^{\delta + p - 1} f(z)}{D^{\delta + p} f(z)} .$$

This can be seen directly by substituting $p(z)$ in the differential equation and using the identity

$$z(D^{\delta + p - 1} f(z))' = (\delta + p)\, D^{\delta + p} f(z) - \delta\, D^{\delta + p - 1} f(z).$$

Of course, by using the above identity, we get

$$\frac{zp'(z)}{p(z)} = -\frac{z(D^{\delta + p} f(z))'}{D^{\delta + p} f(z)} + \frac{z(D^{\delta + p - 1} f(z))'}{D^{\delta + p - 1} f(z)}$$

$$= -(\delta + p + 1)\frac{D^{\delta + p + 1} f(z)}{D^{\delta + p} f(z)} + (\delta + p)\frac{D^{\delta + p + 1} f(z)}{D^{\delta + p - 1} f(z)} + 1.$$

Therefore,

$$B(z) + \frac{\alpha}{\delta + p + 1 - \alpha}\,\frac{zp'(z)}{p(z)} = \frac{1}{p(z)}$$

which shows that $p(z)$ is the solution of the differential equation. Since $\delta + p + 1 > \alpha > 0$, Theorem A shows that the function $p(z)$ is in $H[1, n]$ and $p(z) \neq 0$ and $p(z) \prec 1/q(z)$. This completes the proof.

The proof of Theorem 2.4 is similar.

Corollary 2.3 (Miller and Mocanu[3]) — Let $q(z)$ be univalent in U with $q(0) = 1$ and $q(z) \neq 0$. Set

$$h'(z) = q(z) + \frac{zq'(z)}{q(z)} ,$$

and suppose that the conditions (a) and (b) or (a) and (b′) of Theorem A are satisfied by $q(z)$ and $h(z)$. If $k(z)$ is as in Theorem 2.4 and $f \in A$ satisfies

$$\frac{zf''(z)}{f'(z)} \prec \frac{zk''(z)}{k'(z)} ,$$

then

$$\frac{zf'(z)}{f(z)} \text{ is analytic and } \frac{zf'(z)}{f(z)} \prec \frac{zk'(z)}{k(z)} .$$

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