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ON ĆIRIĆ'S CONTRACTION OPERATOR AND FIXED POINTS

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In this paper Ćirić's contraction operators have been studied in comparison with generalized Kannan type operators with illustrative examples. Some fixed point theorems have been proved for Ćirić contraction operator in a metric space.

Over a metric space X a contraction operator is always a Ćirić operator or a generalized Kannan type operator, however the converse is false. In general, a Ćirić operator over X may not possesses a fixed point; neither does a generalized Kannan type operator. In this paper we have shown by examples that these two types of operators are independent in the sense that neither implies the other. Some fixed point theorems have been proved for a Ćirić operator that is also a generalized Kannan type operator. Continuity of fixed points of such operators has also been proved in this paper. Strength of the hypothesis made in theorems has been examined by illustrative examples.

Let (X, d) be a metric space.

Definition — A mapping $T : X \to X$ is said to be Ćirić operator if there are non-negative real valued functions q and δ over $X \times X$ satisfying $d(T^n(x), T^n(y)) \leq q^n(x, y)\, \delta(x, y)$, $n = 1, 2, \ldots$ for all $x, y \in X$ where $q(x, y) < 1$ with $\sup_{x, y \in X} q(x, y) = 1$.

Let the set of all Ćirić operators be denoted by $Ci(X)$.

Theorem 1 — Let (X, d) be a complete metric space and $T \in Ci(X)$ satisfying

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y)$$

$$+ \gamma \max \{d(x, T(y)), d(y, T(x))\} \text{ for all } x, y \in X$$

where $\alpha, \beta, \gamma \geq 0$ are such that $\max \{\alpha, \beta\} + \gamma < 1$. Then T has a unique fixed point in X.

PROOF : Let $x_0 \in X$. Then for positive integers m and n

$$d(T^m(x_0), T^n(x_0)) = d(T(T^{m-1}(x_0)), T(T^{n-1}(x_0)))$$

$$\leq \alpha\, [d(T^{m-1}(x_0), T^m(x_0)) + d(T^{n-1}(x_0), T^n(x_0))]$$

$$+ \beta d(T^{m-1}(x_0), T^{n-1}(x_0))$$

$$+ \gamma \max\, [d(T^{m-1}(x_0), T^n(x_0)), d(T^{n-1}(x_0), T^m(x_0))]$$

$$\leq \alpha\, [d(T^{m-1}(x_0), T^m(x_0)) + d(T^{n-1}(x_0), T^n(x_0))]$$

$$+ \beta d(T^{m-1}(x_0), T^{n-1}(x_0))$$

$$+ \gamma \max\, [d(T^{m-1}(x_0), T^m(x_0)) + d(T^m(x_0)), T^n(x_0)), d(T^{n-1}(x_0), T^n(x_0))$$

$$+ d(T^n(x_0), T^m(x_0))] \leq \alpha\, [d(T^{m-1}(x_0), T^m(x_0))$$

$$+ d(T^{n-1}(x_0), T^n(x_0))] + \beta\, [d(T^{m-1}(x_0), T^m(x_0))$$

$$+ d(T^m(x_0), T^n(x_0)) + d(T^n(x_0), T^{n-1}(x_0))]$$

$$+ \gamma\, [d(T^{m-1}(x_0), T^m(x_0)) + d(T^m(x_0), T^n(x_0)) + d(T^{n-1}(x_0), T^n(x_0))]$$

$$\leq (2 \max\{\alpha, \beta\} + \gamma)\; [d(T^{m-1}(x_0), T^m(x_0)) + d(T^{n-1}(x_0), T^n(x_0))]$$

$$+ (\beta + \gamma)\, d(T^m(x_0), T^n(x_0))$$

$$\therefore \qquad d(T^m(x_0), T^n(x_0)) \leq \frac{2 \max\{\alpha, \beta\} + \gamma}{1 - \beta - \gamma}$$

$$\times [d(T^{m-1}(x_0), T^m(x_0)) + (T^{n-1}(x_0), T^n(x_0))]$$

$$\leq \frac{2 \max\{\alpha, \beta\} + \gamma}{1 - \beta - \gamma}\, [q^{m-1}(x_0, T(x_0))$$

$$+ q^{n-1}(x_0, T(x_0))]\, \delta(x_0, T(x_0))$$

$\to 0$ as $m, n \to \infty$.

So, $\{T^n(x_0)\}$ is Cauchy, and let $\lim_n T^n(x_0) = u$ for some $u \in X$.

Now $$d(T^n(x_0), T(u) \leq \alpha[d(T^{n-1}(x_0) \cdot T^n(x_0))$$

$$+ d(u, T(u))] + \beta d\,(T^{n-1}(x_0), u)$$

$$+ \gamma \max\, \{d(T^{n-1}(x_0), T(u), d(u, T^n(x_0))\}.$$

As $n \to \infty$, this gives

$$d(u, T(u)) \leq (\alpha + \gamma)\, d(u, T(u)) \text{ and hence } u = T(u).$$

For uniqueness of u, let $v = T(v)$ for $v \in X$.

Then $$d(u, v) = d(T(u), T(v)) = d(T^n(u), T^n(v))$$

$$\leq q^n(u, v)\, \delta(u, v) \to 0 \text{ as } n \to \infty. \text{ Hence } u = v.$$

Theorem 2 — Let X be a metric space and let $T \in Ci(X)$ satisfying

$$d(T(x), T(y)) \le \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) + \gamma \max [d(x, T(y)), d(y, T(x))] \text{ for all } x, y \in X$$

where $\alpha, \beta, \gamma \ge 0$ are such that $\max \{\alpha, \beta\} + \gamma < 1$.

If the sequence of iterates $\{T^n(x_0)\}$ for some $x_0 \in X$ has a subsequence $\{T^{n_k}(x_0)\}$ with $\lim_k T^{n_k}(x_0) = u \in X$. Then u is the unique fixed point and $\lim_n T^n(x_0) = u$.

PROOF : Let $\lim_k T^{n_k}(x_0) = u$ and $u \in X$.

Then
$$d(u, T(u)) \le d(u, T^{n_k+1}(x_0)) + d(T^{n_k+1}(x_0), T(u))$$

$$\le d(u, T^{n_k+1}(x_0)) + \alpha [d(T^{n_k}(x_0), T^{n_k+1}(x_0)) + d(u, T(u))] + \beta d[T^{n_k}(x_0), u) + \gamma \max [d(T^{n_k}(x_0), T(u)), d(u, T^{n_k+1}(x_0))]$$

$$\le d(u, T^{n_k}(x_0)) + d(T^{n_k}(x_0), T^{n_k+1}(x_0)) + \alpha [d(T^{n_k}(x_0), T^{n_k+1}(x_0)) + d(u, T(u))] + \beta d(T^{n_k}(x_0), u) + \gamma \max [d(T^{n_k}(x_0), T(u)), d(u, T^{n_k}(x_0)) + d(T^{n_k}(x_0), T^{n_k+1}(x_0))]$$

$$\le d(u, T^{n_k}(x_0)) + q^{n_k}(x_0, T(x_0)) \cdot \delta(x_0, T(x_0)) + \alpha q^{n_k}(x_0, T(x_0)) \cdot \delta(x_0, T(x_0)) + \alpha d(u, T(u)) + \beta d(T^{n_k}(x_0), u) + \gamma \max [d(T^{n_k}(x_0), T(u)), d(u, T^{n_k}(x_0)) + q^{n_k}(x_0, T(x_0)) \delta(x, T(x_0))]$$

$$\to \alpha d(u, T(u)) + \gamma d(u, T(u)) \text{ as } k \to \infty \text{ i.e. } (\alpha + \gamma) d(u, T(u)) \text{ as } k \to \infty.$$

This gives $u = T(u)$, and uniqueness of u is also now clear.

Finally, $d(u, T^n(x_0)) = d(T^n(u), T^n(x_0)) \le q^n(u, x_0). \delta(u, x_0) \to 0$ as $n \to \infty$.

Example 1 shows that condition that T is a Ćirić operator in Theorem 1 cannot be relaxed.

Example 1 — Take $X = \{a, b\}$, $a \ne b$ with a matric d as $d(a, b) = d(b, a) = 1$ and $d(a, a) = d(b, b) = 0$. Let $T : X \to X$ be defined as $T(a) = b$ and $T(b) = a$. Taking $\alpha = \frac{1}{2}$, $\beta = \gamma = 0$ we find

$$d(T(x), T(y)) \le \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) + \gamma \max[d(x, T(y)), d(y, T(x))]$$

where $\max\{\alpha, \beta\} + \gamma < 1$ and $x, y \in X$. Further $d(T^n(a), T^n(b)) = 1 > q^n(a, b). \delta(a, b)$ for large n where q and δ are non-negative real valued function over $X \times X$ with $q(x, y) < 1$ $(x \neq y)$ and $\sup_{x, y \in X} q(x, y) = 1$. Hence T. is not a Ćirić operator, and T has no fixed point in X either.

Example 2 shows that condition that T satisfies

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) + \gamma \max [d(x, T(y)), d(y, T(x))]$$

where $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha, \beta\} + \gamma < 1$, in Theorem 1 can neither be relaxed.

Example 2 — Take $X = [0, \infty)$ with usual metric and $T : X \to X$ by

$$T(x) = \frac{1}{3} \text{ for } 0 \leq x \leq 1 \text{ except } x = \frac{1}{80}, \frac{1}{3^i} \ (i = 1, 2, \ldots)$$

$$= 0 \text{ for } x = \frac{1}{80} \text{ and } x > 1$$

and $$T\left(\frac{1}{3^i}\right) = \frac{1}{3^{i+1}}; \quad i \geq 1.$$

Let us take $q(x, y) = 1 - \dfrac{1}{xy + 2}$, and $\delta(x, y) = 3 + xy$ for $x, y \in X$. Then $q(x, y) < 1$ with $\sup_{x, y \in X} q(x, y) = 1$. Then by routine exercise we verify that $T \in C_i(X)$.

Now for $x = \dfrac{1}{4}$ and $y = \dfrac{1}{80}$ we may also verify that T does not satisfy the condition

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) + \gamma \max [d(x, T(y)), d(y, T(x))] \text{ for } \alpha, \beta, \gamma \geq 0$$

with $\max\{\alpha, \beta\} + \gamma < 1$.

Theorem 3 — Let (X, d) be a complete metric space and let $T_j \in C_i(X)$ satisfy

$$d(T_j(x), T_j(y)) \leq \alpha[d(x, T_j(x)) + d(y, T_j(y))] + \beta d(x, y) + \gamma \max [d(x, T_j(y)), d(y, T_j(x))]$$

for all, $x, y \in X$ when $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha, \beta\} + \gamma < 1$ with fixed points u_j $(j = 1, 2, \ldots)$. Suppose

$$T^n(x) = \lim_{j \to \infty} T_j^n(x) \quad \text{for all } x \in X.$$

Then T has the unqiue fixed point u in X if and only if

$$u = \lim_{j} u_j.$$

PROOF : $T^n(x) = \lim_{j \to \infty} T_j^n(x)$ for $x \in X$.

For $x, y \in X$, we have $d(T_j(x), T_j(y)) \le \alpha[d(x, T_j(x)) + d(y, T_j(y))]$

$$+ \beta d(x, y) + \gamma \max [d(x, T_j(y)), d(y, T_j(x))]$$

where $\alpha, \beta, \gamma \ge 0$ with $\max \{\alpha, \beta\} + \gamma < 1$. ... (*)

For a positive integer n,

$$d(T^n(x), T^n(y)) \le d(T^n(x), T_j^n(x)) + d(T_j^n(x), T_j^n(y)) + d(T_j^n(y), T^n(y))$$

$$\le d(T^n(x), T_j^n(x)) + q^n(x, y).\, \delta(x, y) + d(T^n(y), T_j^n(y)) \quad \text{... (**)}$$

As $j \to \infty$ from (*) and (**) one finds $T \in C_i(X)$ satisfying

$$d(T(x), T(y)) \le \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y)$$
$$+ \gamma \max [d(x, T(y)), d(y, T(x))].$$

If u is the unique fixed point of T and $u_j = T_j(u_j)$; $j = 1, 2, \ldots$ (Theorem 1), then we have

$$d(u, u_j) = d(T(u), T_j(u_j))$$
$$\le d(T(u), T_j(u)) + d(T_j(u), T_j(u_j))$$
$$\le d(T(u), T_j(u)) + \alpha[d(u, T_j(u)) + d(u_j, T_j(u_j))]$$
$$+ \beta d(u, u_j) + \gamma \max [d(u_j, T_j(u_j)), d(u_j, T_j(u))]$$
$$\le d(T(u), T_j(u)) + \alpha d(T(u), T_j(u)) + \beta d(u, u_j)$$
$$+ \gamma \max [d(u, u_j), d(u_j, u) + d(u, T_j(u))]$$
$$\le d(T(u), T_j(u)) + \alpha d(T(u), T_j(u)) + \beta d(u, u_j)$$
$$+ \gamma[d(u, u_j) + d(u, T_j(u))]$$

$$\therefore \quad d(u, u_j) \le \frac{1 + \alpha + \gamma}{1 - \beta - \gamma} d(T(u), T_j(u))$$

$\to 0$ as $j \to \infty$.

Conversely, let $u = \lim_{j} u_j$ then

$$d(T_j(u), T_j(u_j)) \le \alpha [d(u, T_j(u)) + d(u_j, T_j(u_j))]$$
$$\beta d(u, u_j) + \gamma \max [d(u, T_j(u_j)), d(u_j, T_j(u))]$$

i.e., $$d(T_j(u), u_j) \le \alpha d(u, T_j(u)) + \beta d(u, u_j) + \gamma \max [d(u, u_j), d(u_j, T_j(u))]$$

$$\leq \alpha d(u, T_j(u)) + \beta d(u, u_j) + \gamma[d(u, u_j) + d(u_j, T_j(u))]$$

$\therefore$ $$(1-\gamma)\, d(T_j(u), u_j)) \leq \alpha d(u, T_j(u)) + (\beta+\gamma)\, d(u, u_j)$$

i.e. $$d(T_j(u), u_j) - \frac{\alpha}{1-\gamma}\, d(u, T_j(u)) \leq \frac{\beta+\gamma}{1-\gamma}\, d(u, u_j)$$

i.e. $$\left(1 - \frac{\alpha}{1-\gamma}\right) d(u, T(u)) \to 0 \text{ as } j \to \infty.$$

i.e. $$\frac{1-(\alpha+\gamma)}{1-\gamma}\, d(u, T(u)) = 0$$

i.e. $d(u, T(u)) = 0$. Hence $T(u) = u$.

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