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ON A THEOREM OF ROBINSON FOR NON-ARCHIMEDIAN BANACH SPACES

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Non-archimedian analogue of Robinson's's[3] theorem is obtained.

§1. After the extension of Kojima-Schur theorem for matrix transformations over nontrivially non-archimedian valued field F by Monna[2], the author[4, 5] had studied the other types of summability matrices over F. The other concepts like the spaces of almost convergent sequences and matrix transformations were also studied in the non-archimedian context (Nanda[6]). Quite recently, attempts have been made as in Komal[1] to study the properties of operators like multiplication operators on sequence spaces over F. The object of this paper is to obtain the non-archimedian operator analogue of Kojima-Schur theorem for Banach spaces due to Robinson[3].

§2. Let K be a field complete with respect to a non-archimedian valuation. Let X be a non-archimedian Banach space. Let $B(X)$ denote the space of bounded linear operators on X into X. Throughout the paper $A = (A_{np})$ where $n, p = 1, 2, 3, \ldots$ is an infinite matrix such that $A_{np} \in B(X)$ for all n and p. $C(X)$ denotes the space of X-valued convergent sequences $x = (x_n)$ with $\| x \| = \sup_{n \geq 1} \| x_n \|$. We know that $C(X)$ is a non-archimedian Banach space. If (T_n) is a sequence in $B(X)$, then the group norm of the sequence is defined as

$$\| (T_n) \| = \sup_{n \geq 1} \| T_n x_n \| \text{ for each}$$

$$x_n \in X \text{ with } \| x_n \| \leq 1.$$

We shall now consider the transformations of the type

$$y_n = \sum_{p=l}^{\infty} A_{np} x_p, n = 1, 2, 3, \ldots$$

Then $A = (A_{np})$ is said to be convergence preserving if (y_n) converges whenever (x_n) converges in the norm. A is said to be regular if $x_p \to s$ as $p \to \infty$ implies $y_n \to s$

as $n \to \infty$. Before establishing Robinson's[3] theorem for non-archimedian Banach space, we shall prove the following basic theorem needed in the proof of the theorem.

Theorem 1 — Let (T_n) be a sequence in $B(X)$. Then the series $\sum T_n x_n$ converges for every convergent sequence (x_n) in X if and only if for each $x \in X$, $\lim_{n \to \infty} T_n x = 0$.

PROOF : The condition is sufficient.

Since $\lim_{n \to \infty} T_n x = 0$ for all $x \in X$, we have

$$\operatorname{Sup}_n \| T_n x \| < \infty \quad \text{for each} \quad x \in X.$$

Hence by the uniform boundedness principle in the non-archimedian case of Monna[2], we get

$$\operatorname{Sup}_{n \geq 1} \| T_n \| \leq M \text{ for all } n. \qquad \text{... (1.1)}$$

Case (i) — Let (x_n) be a null sequence. Then given $\varepsilon > 0$ there exists a n_0 such that

$$\| x_n \| < \varepsilon \text{ for all } n \geq n_0 \qquad \text{... (1.2)}$$

Taking $y_m = \sum_{n=1}^{m} T_n x_n$, we get

$$\| y_{m+p} - y_m \| = \left\| \sum_{y=m+1}^{m+p} T_n x_n \right\| \leq \operatorname{Max}_{m+1 \leq n \leq m+p} \| T_n \| \| x_n \|.$$

Choosing $m > n_0$, we get

$$\| y_{m+p} - y_m \| < M \varepsilon \text{ for } p = 1, 2, 3, \ldots \text{ by (1.1) and (1.2).}$$

Hence (y_n) converges which proves that $\sum_{n=1}^{\infty} T_n x_n$ converges.

Case (ii) — Let (y_n) converge to s in X.

Then we can write

$$\sum_{n=1}^{\infty} T_n x_n = \sum_{n=1}^{\infty} T_n (x_n - s) + \sum_{n=1}^{\infty} T_n s = S_1 + S_2 \text{ (say).}$$

Since $(x_n - s)$ is a null sequence, S_1 converges as in Case (i).

Since $T_n s \to 0$ as $n \to \infty$, S_2 converges. Thus we have proved that $\sum T_n x_n$ is convergent and so the condition is sufficient.

The condition is necessary. Let us suppose that $\sum T_n x_n$ converges for every convergent sequence (x_n). Then for each x, $\sum_{n=1}^{\infty} T_n x$ converges. Since X is complete, for each $x \in X$, $\lim_{n \to \infty} T_n x = 0$. Hence the condition is necessary.

Theorem 2 — If $T_n x \to 0$ for every $x \in X$, then (T_n) is collectively bounded.

PROOF : If (T_n) is not collectively bounded, then $T_n x_n \to \infty$ through a subsequence of values of n. Since K is a non-trivially valued field, there exists a sequence (λ_n) in K such that $|\lambda_n| \to \infty$ as $n \to \infty$. Therefore there exists a subsequence (λ_{n_p}) of (λ_n) such that

$$\| T_{n_p}(x_{n_p}) \| > |\lambda_{n_p}| \text{ for all } \|x_p\| < 1. \qquad \text{... (2.1)}$$

Let us define a sequence (x_n) as follows

$$x_n = 0 \text{ for all } n \neq n_p \text{ and } x_{n_p} = \frac{x}{\lambda_{n_p}} \text{ where } \| x \| = 1.$$

From the definition of the sequence (x_n), $\|x_{n_p}\| < 1$ for large values of p.

Now $T_{n_p}(x_p) = T_{n_p}\left(\frac{x}{\lambda_{n_p}}\right) > 1$ from (2.1).

Thus $(T_n x_n)$ does not tend to zero, contradicting the hypothesis. Hence (T_n) is collectively bounded.

§3. Having established necessary preliminaries, we are now in a position to prove Robinson's theorem for an infinite matrix of bounded linear operators on a non-archimedean Banach space X into itself. Let $A = (A_{np})$ be an infinite matrix whose entries belong to $B(X)$ and let I denote the identity operator on X. If (x_p) is a convergent sequence in X, then the transformed sequence (y_n) is defined by

$$y_n = \sum_{p=1}^{\infty} A_{np} x_p, \; n = 1, 2, 3, \ldots . \qquad \text{... (3.1)}$$

Theorem 3 — For the matrix $A = (A_{np})$ is to be regular, it is necessary and sufficient that for each $x \in X$,

$$\lim_{n \to \infty} A_{np} x = 0 \text{ for each fixed } p \qquad \text{... (3.2)}$$

$$\lim_{n \to \infty} \sum_{p=1}^{\infty} A_{np} \, x = Ix \qquad \text{... (3.3)}$$

$$\lim_{p \to \infty} A_{np} x = 0 \text{ for each fixed } n \quad \ldots (3.4)$$

$$\sup_{\substack{1 \le p < \infty \\ 1 \le n < \infty}} \| A_{np} \| \le M \text{ where } M \text{ is a constant} \quad \ldots (3.5)$$

independent of n and p.

PROOF : We shall prove that the conditions are sufficient. Let (x_p) converge to a limit s. We have to show that under the condition (3.2)-(3.5), (y_n) given by (3.1) will also converge to s.

If $x_p \to s$ as $p \to \infty$, given $\varepsilon > 0$, we can choose a p_0 such that

$$\| x_p - s \| < \varepsilon \text{ for all } p \ge p_0. \quad \ldots (3.6)$$

By (3.3) and (3.2), we can choose a n_0 such that

$$\left\| \left(I - \sum_{p=1}^{\infty} A_{np} \right) s \right\| < \varepsilon \text{ for all } n \ge n_0 \quad \ldots (3.7)$$

$$\| A_{np} (x_p - s) \| < \frac{\varepsilon}{p_0} \text{ for all } n \ge n_0 \text{ and } p \le p_0. \quad \ldots (3.8)$$

Now $$\| y_n - s \| = \left\| \sum_{p=1}^{p_0} A_{np} (s_p - s) + \sum_{p_0+1}^{\infty} A_{np} (s_p - s) - \left(I - \sum_{p=1}^{\infty} A_{np} \right) s \right\| .$$

Since we are considering the non-archimedian norm.

$\| y_n - s \|$

$$\le \text{Max} \left\{ \left\| \sum_{p=1}^{p_0} A_{np} (s_p - s) \right\| , \left\| \sum_{p_0+1}^{\infty} A_{np} (s_p - s) \right\| , \left\| \left(I - \sum_{p=1}^{\infty} A_{np} \right) s \right\| \right\} . \quad \ldots (3.9)$$

By using (3.5) and (3.6), we have

$$\left\| \sum_{p_0+1}^{\infty} A_{np} (s_p - s) \right\| \le \sup_{p \ge p_0} \| A_{np} \| \, \| (s_p - s) \| < M \varepsilon. \quad \ldots (3.10)$$

By using (3.8), we have

$$\left\| \sum_{p=1}^{p_0} A_{np} (s_p - s) \right\| \le \sup_{1 \le p < p_0} \| A_{np} (s_p - s) \| < \frac{\varepsilon}{p_0} p_0 ,$$

for all $n \ge n_0$. ... (3.11)

Using (3.7), (3.10) and (3.11) in (3.9), we get

$$\| y_n - s \| < \text{Max}\left\{\frac{\varepsilon}{p_0}, M\varepsilon, \varepsilon\right\} = M\varepsilon \text{ for all } n \geq n_0.$$

Hence $y_n \to s$ as $n \to \infty$. This proves that the conditions are sufficient.

To prove the necessity of the conditions, we proceed as follows.

Let us consider the sequence (x_n) where $x_p = x$ and $x_j = 0$ for $j \neq p$. Then we have $y_n = A_{np} x$. Now the sequence $x_p \to 0$ as $p \to \infty$. Therefore the sequence $y_n = A_{np} x \to 0$ as $n \to \infty$ for each fixed p. Hence we get the condition (3.2).

Let us take the sequence (x_p) where $x_p = x$ for all p. Then $x_p \to x$ as $p \to \infty$. Hence $y_n = \sum_{p=1}^{\infty} A_{np} x \to x$ as $n \to \infty$ for all x so that we have condition (3.3).

Further by Theorem 1, we get $\lim_{p \to \infty} A_{np} x = 0$ for each $x \in X$ for each fixed n which proves (3.4).

Now we shall proceed to establish the necessity of the condition (3.5) as follows.

For this purpose, we assume that the condition (3.5) is not satisfied and construct a sequence (x_n) tending to zero whose A-transform does not tend to zero.

Since the condition (3.5) is not satisfied

$$\sup_{1 \leq p < \infty} \| A_{np} \| \to \infty \text{ as } n \to \infty \qquad \text{... (3.12)}$$

through a subsequence of values of n. Hence there exists a n_1 such that

$$\sup_{1 \leq p < \infty} \| A_{n_1 p} \| > \frac{1}{\lambda} \text{ where } |z| = \lambda < 1$$

where z is some element of K for which $| z | = \lambda < 1$.
Since $\| A_{n_1 p} \| = \sup_{\|x\| \leq 1} \| A_{n_1 p} x \|$, we have from the above,

$$\sup_{1 \leq p < \infty} \| A_{n_1 p} x \| > \frac{1}{\lambda} \text{ where } \| x \| \leq 1. \qquad \text{... (3.13)}$$

Since (3.4) is true for each n, given $\lambda < 1$, we can find a p_{n_1} for $n = n_1$ such that

$$\sup_{p_{n_1} + 1 \leq p < \infty} \| A_{n_1 p} x \| < \lambda \text{ for all } p \geq p_{n_1} \text{ and for all } x \in X. \qquad \text{... (3.14)}$$

From (3.13) and (3.14), we get

$$\sup_{1 \leq p \leq p_{n_1}} \| A_{n_1 p} x \| > \frac{1}{\lambda} \text{ for every } x \in X. \qquad \text{... (3.15)}$$

Hence there exists a p_1 in $1 \leq p \leq p_{n_1}$ such that

$$\| A_{n_1 p_1} x \| > \frac{1}{\lambda} \text{ for all } x \in X. \qquad \text{... (3.16)}$$

Let us define a sequence (x_p) as follows.

$$x_p = \begin{cases} z^{n_1} x & \text{for } p = p_1 \text{ and } \|x\| = 1 \\ 0 & \text{for all } p \text{ in } 1 \le p \le p_{n_1} \text{ and } p \ne p_1. \end{cases} \qquad \text{... (3.17)}$$

Corresponding to n_1, we can rewrite (3.1) as

$$\sum_{p=1}^{p_{n_1}+1} A_{n_1 p} x_p = y_{n_1} - \sum_{p_{n_1}+1}^{\infty} A_{n_1 p} x_p.$$

Using (3.17) in the above, we get

$$\| A_{n_1 p_1} zx \| \le \text{Max} \left\{ \| y_{n_1} \|, \left\| \sum_{p_{n_1}+1}^{\infty} A_{n_1 p} x_p \right\| \right\}. \qquad \text{... (3.18)}$$

Using (3.14), we get

$$\text{But } \left\| \sum_{p_{n_1}+1}^{\infty} A_{n_1 p} x_p \right\| \le \sup_{p_{n_1}+1 \le p < \infty} \| A_{n_1 p} x_p \| < \lambda. \qquad \text{... (3.19)}$$

Using (3.16), and (3.19) in (3.18), we get

$$\frac{1}{\lambda} \le \text{Max} \{ \| y_{n_1} \|, \lambda \} \text{ so that } \| y_{n_1} \| > \frac{1}{\lambda}.$$

By Theorem 2, the condition (3.2) implies that (A_{np}) is collectively bounded. This means that

$$\| A_{np} x \| \le M_p \text{ for all } n, \text{ fixed } p \text{ and for all } x \in X. \qquad \text{... (3.20)}$$

Using (3.20), we get

$$\left\| \sum_{p=1}^{p_{n_1}} A_{np} x_p \right\| \le \max_{1 \le p \le p_{n_1}} \| A_{np} x_p \| \text{ for all } n$$

$$\le \text{Max} \{ M_1, M_2, \dots, M_p \}.$$

If $Q_{n_1} = \text{Max} \{1, M_1, M_2, \dots, M_p\}$, then we get

$$\sup_{1 \le p \le p_{n_1}} \| A_{np} x_p \| < Q_{n_1}. \qquad \text{... (3.21)}$$

Now we can choose $n_2 > n_1$ such that

$$\operatorname*{Sup}_{1 \le p < \infty} \|A_{n_2 p}\| > \frac{Q_{n_1}}{\lambda^2}$$

which implies

$$\operatorname*{Sup}_{1 \le p < \infty} \|A_{n_2 p} x\| > \frac{Q_{n_1}}{\lambda^2} \text{ for } \|x\| \le 1. \qquad \dots (3.22)$$

From (3.4), there exists a $p_{n_2} > p_{n_1}$ such that

$$\operatorname*{Sup}_{p_{n_2}+1 \le p < \infty} \|A_{n_2 p} x\| < \lambda. \qquad \dots (3.23)$$

From (3.22) and (3.23), we get

$$\operatorname*{Sup}_{1 \le p \le p_{n_2}} \|A_{n_2 p} x\| > \frac{Q_{n_1}}{\lambda^2}.$$

Therefore there exists a p_2 in $1 \le p \le p_{n_2}$ such that

$$\|A_{n_2 p_2} x\| > \frac{Q_{n_1}}{\lambda^2}. \qquad \dots (3.24)$$

For the n_2 chosen, we have from (3.21),

$$\operatorname*{Sup}_{1 \le p \le p_{n_1}} \|A_{n_2 p} x\| < Q_{n_1} \text{ for all } x \in X. \qquad \dots (3.25)$$

Therefore p_2 chosen above in (3.24) exceeds p_{n_1}.

Having chosen n_2 as above, define x_p as follows

$$x_p = \begin{cases} z^{n_2} x \text{ for } p = p_2 \text{ such that } p_{n_1} + 1 \le p_2 \le p_{n_2} \\ \qquad\qquad\qquad\qquad \text{and } \|x\| = 1 \\ 0 \text{ for all } p \text{ in } p_{n_1} + 1 \le p \le p_{n_2},\ p \ne p_2. \end{cases} \qquad \dots (3.26)$$

Corresponding to n_2, we can rewrite (3.1) as

$$\sum_{p_{n_1}+1}^{p_{n_2}} A_{n_2 p} x_p = y_{n_2} - \sum_{p=1}^{p_{n_1}} A_{n_2 p} x_p - \sum_{p_{n_2}+1}^{\infty} A_{n_2 p} x_p.$$

Hence we get from the above step

$$\left\| \sum_{p_{n_1}+1}^{p_{n_2}} A_{n_2 p} x_p \right\|$$

$$\leq \text{Max}\left\{ \|y_{n_2}\|, \left\| \sum_{p=1}^{p_{n_1}} A_{n_2 p} x_p \right\|, \left\| \sum_{p_{n_2}+1}^{\infty} A_{n_2 p} x_p \right\| \right\}. \qquad \text{... (3.27)}$$

From (3.25), we get

$$\left\| \sum_{p=1}^{p_{n_1}} A_{n_2 p} x_p \right\| < \sup_{1 \leq p \leq p_{n_1}} \|A_{n_2 p} x_p\| < Q_{n_1}. \qquad \text{... (3.28)}$$

From (3.23), we get

$$\left\| \sum_{p_{n_2}+1}^{\infty} A_{n_2 p} x_p \right\| < \text{Sup}\, \|A_{n_2 p} x_p\| \leq \lambda. \qquad \text{... (3.29)}$$

Using (3.26), (3.28) and (3.29) in (3.27) we get

$$\frac{Q_{n_1}}{\lambda^2} < \text{Max } \{\|y_{n_2}\|, Q_{n_1}, \lambda\} \text{ so that } \|y_{n_2}\| > \frac{Q_{n_1}}{\lambda^2}.$$

Since $Q_{n_1} > 1$, we have $\|y_{n_2}\| > \frac{1}{\lambda^2}$

Proceeding in this manner, let us suppose that n_{k-1}, p_{k-1} have been chosen. By Theorem 2 and condition 3.2, we get

$$\left\| \sum_{p=1}^{p_{n_{k-1}}} A_{np} x_p \right\| \leq \sup_{1 \leq p \leq p_{n_{k-1}}} \|A_{np} x_p\| \text{ for all } n \text{ and for all } x_p \in X$$

$$\leq \text{Max } [M_1, M_2, ..., M_{p_{n_{k-1}}}]$$

$$\leq Q_{n_{k-1}} \text{ where}$$

$$Q_{n_{k-1}} = \text{Max } [1, M_1, M_2, ..., M_{p_{n_{k-1}}}].$$

$$\sup_{1 \leq p \leq p_{n_{k-1}}} \|A_{np} x_p\| < Q_{n_{k-1}}. \qquad \text{... (3.30)}$$

Now choose $n_k > n_{k-1}$ such that

$$\sup_{1 \leq p < \infty} \|A_{n_k p}\| > Q_{n_{k-1}} | \lambda^k$$

which implies

$$\sup_{1 \le p < \infty} \| A_{n_k p} x \| > Q_{n_{k-1}} | \lambda^k \text{ for all } \| x \| \le 1. \quad \text{... (3.31)}$$

From (3.4) there exists $p_{n_k} > p_{n_{k-1}}$ such that

$$\sup_{p_{n_k}+1 < p < \infty} \| A_{n_k p} x \| < \lambda. \quad \text{... (3.32)}$$

From (3.31) and (3.32), we get

$$\sup_{1 \le p < p_{n_k}} \| A_{n_k p} x \| > Q_{n_{k-1}} / \lambda^k. \quad \text{... (3.33)}$$

Therefore there exists a p_k in $1 < p < p_{n_k}$ for which

$$\| A_{n_k p_k} x \| > Q_{n_{k-1}} / \lambda^k. \quad \text{... (3.34)}$$

(3.30) is true for $n = n_k$ also. So we get

$$\sup_{1 < p < p_{n_{k-1}}} \| A_{n_k p} x \| < Q_{n_{k-1}}. \quad \text{... (3.35)}$$

Therefore p_k chosen in (3.34) exceeds $p_{n_{k-1}}$.

Now choose x_p for p in $p_{n_{k-1}} + 1 < p < p_{n_k}$ as follows

$$x_p = \begin{cases} Z^{n_k} x \text{ for } p = p_k \text{ and } \| x \| = 1 \\ 0 \text{ for all } p \text{ in } p_{n_{k-1}} + 1 \le p \le p_{n_k} \text{ and } p \ne p_k. \end{cases} \quad \text{... (3.36)}$$

Corresponding to $n = n_k$, we can rewrite (3.1) as

$$\sum_{p_{n_{k+1}}+1}^{p_{n_k}} A_{n_k p} x_p = y_{n_k} - \sum_{1}^{p_{n_{k-1}}} A_{n_k p} x_p - \sum_{p_{n_k}+1}^{\infty} A_{n_k p} x_p. \quad \text{... (3.37)}$$

From (3.32) we have

$$\left\| \sum_{p_{n_k}+1}^{\infty} A_{n_k p} x \right\| < \lambda. \quad \text{... (3.38)}$$

Making use of (3.34), (3.35), (3.36) and (3.38) in (3.37), we have

$$\frac{Q_{n_k}}{\lambda^k} < \operatorname{Max} \left\{ \| y_{n_k} \|, \, Q_{n_{k-1}}, \lambda \right\}$$

which implies $\| y_{n_k} \| > Q_{n_k}/\lambda^k$.

Since $Q_{n_{k-1}} > 1$, we get $\| y_{n_k} \| > 1/\lambda^k$.

Hence (y_n) tends to infinity as $n \to \infty$ through a subsequence (n_k). This shows that (y_n) does not tend to zero, while (x_p) tends to zero as $Z^p \to 0$ as $p \to \infty$. This contradiction proves the necessity of the condition. Hence the condition (3.5) is necessary. This completes the proof of the theorem.

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