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NUMERICAL STUDY OF THE UNSTEADY FLOW IN FLUID-PARTICLE SUSPENSION FROM AN INFINITE ROTATING DISK

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This paper presents a numerical study of similarity solution of the unsteady flow of the fluid-particle suspension from an infinite rotating disk. The formulation is reduce to a ten coupled first order system of nonlinear ordinary differential equations which are solved numerically by the shooting method using fourth order Runge-Kutta method. It is found that nature of velocity component is similar for both accelerating and decelerating flow, but increases in magnitude as α, parameter characterizing unsteadiness, decreases. Self similar solution is obtained. The results are in good agreement with available results for $\alpha = 0$(steady case). Boundary layer thickness for both the fluid and particle cloud are found to be approximately equal. The effect of unsteadiness can be noted.

INTRODUCTION

The motion of fluid induced by a rotating disk is of special interest in the theory of incompressible flow and hence discussed by several authors, because of the remarkable transformation which reduces the Navier-Stokes equations to set of ordinary differential equations. Karman[1] first studied the steady motion of viscous fluid in the semi-infinite region bounded by a single rotating disk. When disk rotates with time dependent velocities, Karmans transformation is again applicable, but now the Navier-Stokes equations reduce to a set of coupled non-linear ordinary differential equations, which was examined by Pearson[4]. The considerable simplification which resulted from the replacement of the original set of partial differential equation by a set of ordinary differential equation has made an in depth numerical study in this area (Stewartson[3], Batchelor[2], Chawala *et al.*[14] etc).

In the recent years numerical techniques like the shooting method[11, 13], finite difference technique[10] have been widely used for solution of various boundary layer problems and found to be very effective.

The main objective of this paper is to study and present a numerical study of fluid-particle suspension by an infinite rotating disk. The self similar solutions have been obtained numerically by the shooting method[9, 12]. The results have been compared with available results[13] for $\alpha = 0$ and found to be in agreement.

Basic Equations

A half infinite space, $z > 0$, is filled with an incompressible fluid containing small particles of a single size with radius σ. The boundary plane, $z = 0$, is rotating at a time dependent angular velocity $\omega(t)$. Using cylindrical coordinates and taking account of the flow quantities which are independent of the angle θ. The governing equations describing in detail the flow fluid of a two-phase medium are discussed in Schilichting[8]. Denote the radial, angular, and axial velocities of fluid by u, v and w; the mass density by ρ; and designate corresponding quantities associated with the particle cloud by subscript p. The continuity equations for the two phases are

$$\frac{\partial(ru)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0 \qquad \text{... (1)}$$

$$\frac{\partial}{\partial r}(r\rho_p u_p) + \frac{\partial}{\partial z}(r\rho_p u_p) = 0 \qquad \text{... (2)}$$

and the corresponding momentum equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} + \frac{w\partial u}{\partial z} = \frac{-1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\frac{u}{r} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{\rho}F_r \qquad \text{... (3)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r}\frac{w\partial v}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\frac{v}{r} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{1}{\rho}F_o \qquad \text{... (4)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial w}{\partial r} + \frac{w\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{1}{\rho}F_z \qquad \text{... (5)}$$

$$\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial r} - \frac{v_p^2}{r} + w_p\frac{\partial u_p}{\partial z} = \frac{-1}{\rho_p}F_r \qquad \text{... (6)}$$

$$\frac{\partial v_p}{\partial t} + u_p\frac{\partial v_p}{\partial r} - \frac{u_p v_p}{r} + w_p\frac{\partial v_p}{\partial z} = \frac{-1}{\rho_p}F_o \qquad \text{... (7)}$$

$$\frac{\partial w_p}{\partial t} + u_p\frac{\partial w_p}{\partial r} + w_p\frac{\partial w_p}{\partial z} = \frac{-1}{\rho_p}F_z \qquad \text{... (8)}$$

p is the gas pressure and F_r, F_o and F_z are forces in radial, angular and axial direction,

$$F_r = \rho_p(u_p - u)/\tau \qquad \text{... (9)}$$

$$\tau = m/6\pi\mu a \qquad \text{... (10)}$$

$$F_o = \rho_p (v_p - v)\tau \qquad \text{... (11)}$$

$$F_z = \rho_p (w_p - w)/\tau \qquad \text{... (12)}$$

where m is the mass of the single particle, τ has the dimension of time. The boundary conditions of fluid and Particle cloud are :

$$\left.\begin{array}{l} u = 0, \quad v = r\omega(t), \quad w = 0, \quad \text{for} \quad z = 0 \\ u = 0, \quad v = 0, \quad w = 0, \quad u_p = 0, \quad v_p = 0, \quad w_p = w, \\ p = p_\infty, \quad \rho_p = k\rho \quad \text{for} \quad z \to \infty. \end{array}\right\} \qquad \text{... (13)}$$

SELF SIMILAR SOLUTION

To study the solution, we introduce dimensionless variable η defined by

$$\eta = (\omega_0/\nu)^{1/2} (1 - \alpha t^*)^{-1/2} z \qquad \text{... (14)}$$

where α is parameter characterizing unsteadiness. By expressing the fluid velocity component, angular velocity and pressure in the following form :

$$\left.\begin{array}{ll} \omega(t) = \omega_0 (1 - \alpha t^*)^{-1} & ; \; u = \omega_0 t(1 - \alpha t^*)^{-1} F(\eta) \\ v = \omega_0 r(1 - \alpha t^*)^{-1} G(\eta) & ; \; w = (\nu\omega_0)^{1/2} (1 - \alpha t^*)^{-1/2} H(\eta) \\ p = \rho\omega_0\nu (1 - \alpha t^*)^{-1} P(\eta) & ; \; \text{here } t^* = \omega_0 t \end{array}\right\} \qquad \text{... (15)}$$

and expressing the particle velocity and density as

$$\left.\begin{array}{ll} u_p = \omega_0 r (1 - \alpha t^*)^{-1} F_p/\rho_r & ; \; v_p = \omega_0 r (1 - \alpha t^*)^{-1} G_p/\rho_r \\ w_p = (\nu\omega_0)^{1/2} (1 - \alpha t^*)^{-1/2} H_p \rho_r & ; \; \rho_r = Q(\eta) \\ \rho_r = \rho_p/p \;\; (\rho_r \text{ is relative density}) & \end{array}\right\} \qquad \text{... (16)}$$

and substituting in the governing eqns. (1)-(8) we get

$$\alpha\left[F + \frac{\eta F'}{2}\right] + F^2 - G^2 + HF' - F'' - \beta(F_p - QF) = 0 \qquad \text{... (17)}$$

$$\alpha\left[G + \frac{\eta G'}{2}\right] + 2FG + HG' - G'' - \beta(G_p - QG) = 0 \qquad \text{... (18)}$$

$$\alpha\,[H + \eta H'] + HH' + P' - H'' - \beta(H_p - QH) = 0 \qquad \text{... (19)}$$

$$\alpha\left[F_pQ^2+\frac{\eta}{2}F_p'\,Q^2-\frac{\eta}{2}F_p\,Q'Q\right]+QF_p^2-QG_p^2$$

$$+\,(QH_p\,F_p'-H_p\,F_p\,Q')+\beta(F_p\,Q^2-Q^3\,F)\;=\;0 \quad \text{... (20)}$$

$$\alpha\left[G_pQ^2+\frac{\eta}{2}G_p'\,Q^2-\frac{\eta}{2}G_p\,Q'\,Q\right]+2QG_p\,H_p+H_p\,G_p'\,Q$$

$$-\;H_p\,G_p\,Q'+\beta(G_p\,Q^2-GQ^3)\;=\;0 \quad \text{... (21)}$$

$$\frac{\alpha}{2}\left[H_pQ^2+\frac{\eta}{2}H'_p\,Q^2-\frac{\eta}{2}H_p\,Q'\,Q\right]+H_p\,H'_p\,Q-H_p^2\,Q'$$

$$+\,\beta(H_pQ^2-Q^3H)\;=\;0 \quad \text{... (22)}$$

$$2F+H'\;=\;0 \quad \text{... (23)}$$

$$2F_p+H_p'\;=\;0 \quad \text{... (24)}$$

where dashes denote the differentiation with respect to η. The boundary conditions can be obtained from (13) and they are :

$$\left.\begin{array}{llll} F(\eta)=0, & G(\eta)=1, & H(\eta)=0, & \text{for } \eta=0,\\ F(\eta)=0, & G(\eta)=0, & P(\eta)=0, & \text{for } \eta=\infty,\\ F_p(\eta)=0, & G_p(\eta)=0, & H_p(\eta)=kH(\eta) & \text{for } \eta=\infty,\\ Q(\eta)=k & \text{for } \eta=\infty, & & \end{array}\right\} \quad \text{... (25)}$$

β is defined as $\beta=(1-\alpha t^*)/\omega_0\tau$. ... (26)

It may be noted that for $\alpha = 0$, eqns. (17) to (24) reduces to those of steady state and are identical to that of Zung[13].

Numerical Method of Solution

The fundamental problem is to find solution to differential equations (17) to (24) subject to the boundary conditions (25) for the various values of parameter α, β and k. It can be seen that effective order of system of differential equations is ten, whereas ten boundary conditions are to be satisfied. For the purpose of Numerical computation, the differential equations (17) to (24) are written in the form of a system of first order differential equations. The following transformation variables were used.

$$\left.\begin{array}{lllll} Y_1 = F; & Y_2 = F'; & Y_3 = G; & Y_4 = G'; & \\ Y_5 = H; & Y_6 = P; & Y_7 = F_p; & Y_8 = G_p; & Y_9 = H_p; \\ Y_{10} = Q; & Y_{11} = Q', & A = \alpha\eta/2 & & \end{array}\right\} \quad \ldots (27)$$

with these substitutions, the system of :s. (17) to (24) becomes :

$$Y_{11} = (\alpha/2)(Y_9 Y_{10}^2 - 2x Y_7 Y_{10}^2) - 2 Y_9 Y_7 Y_{10}$$
$$+ \beta(Y_9 Y_{10}^2 - Y_{10}^3 Y_5)/(A Y_9 Y_{10} + Y_9^2)$$

$$Y_1' = Y_2 \quad ; \quad Y_2' = \alpha(Y_1 + \eta Y_2/2) + Y_1^2 - Y_3^2 + Y_5 Y_2 - \beta(Y_7 - Y_{10}^2)$$

$$Y_3' = Y_4 \quad ; \quad Y_4' = \alpha(Y_3 + \eta Y_4/2) + 2Y_1 Y_3 + Y_4 Y_5 - \beta(Y_8 - Y_{10} Y_3)$$

$$Y' = -2Y_1 \quad ; \quad Y_6' = -2Y_2 + \beta(Y_9 - Y_{10} Y_5) + 2Y_5 Y_1 - (\alpha/2)(Y_5 - 2\eta Y_1)$$

$$Y_7' = (-\alpha Y_7 Y_{10}^2 + A Y_7 Y_{11} Y_{10} - Y_{10} Y_7^2 + Y_{10} Y_8^2 + Y_9 Y_7 Y_{11}$$
$$+ \beta(Y_7 Y_{10}^2 - Y_{10}^3 Y_1))/(A Y_{10}^2 + Y_{10} Y_9)$$

$$Y_8' = (-\alpha Y_8 Y_{10}^2 + A Y_8 Y_{11} Y_{10} - 2Y_{10} Y_8 Y_9 + Y_9 Y_8 Y_{11}$$
$$- \beta(Y_8 Y_{10}^2 - Y_3 Y_{10}^3))/(A Y_{10}^2 + Y_9 Y_{10})$$

$$Y_9' = -2y_7 \quad ; \quad Y_{10}' = Y_{11}. \quad \ldots (28)$$

The boundary conditions (25) give

$$\left.\begin{array}{llll} Y_1(0) = 0, & Y_3(0) = 1, & Y_5(0) = 0, & \\ Y_1(\infty) = 0, & Y_3(\infty) = 0, & Y_6(\infty) = 0, & \\ Y_7(\infty) = 0, & Y_8(\infty) = 0, & Y_9(\infty) = kY_5(\infty), & Y_{10}(\infty) = k. \end{array}\right\} \quad \ldots (29)$$

Numerical integration was carried out from large value of $\eta = \eta_0$ to $\eta = 0$, missing initial condition are guessed properly. In this case choice of missing initial solution is difficult. An iterative shooting method[9] which uses globally convergent Newton Raphson method is employed. The system is then integrated as an initial value problem, arriving at the other boundary. The guess solution is changed in systematic way until correct starting values are determined. At this stage all boundary conditions are knwon and solution is ready. The shooting method provides a systematic approach for this. Classical fourth order Runge-Kutta method with automatic step size control is used. Solution were obtained for various values of α, β and k. All calculation are done on IBM PC-386 at V.R.C.E., Nagpur.

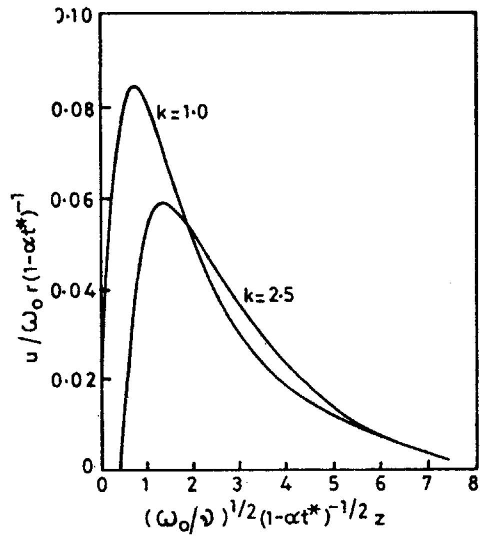

FIG. 1. Dependence of fluid velocities in radial direction upon particle loading (β = 0.5 and α = 1.0).

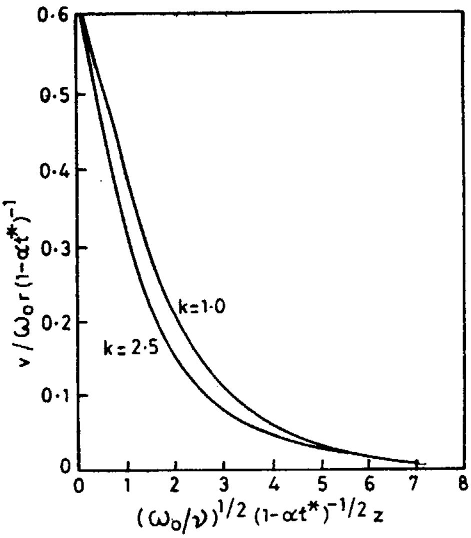

FIG. 2. Dependence of fluid velocities in tangential direction upon particle loading (β = 0.5 and α = 1.0).

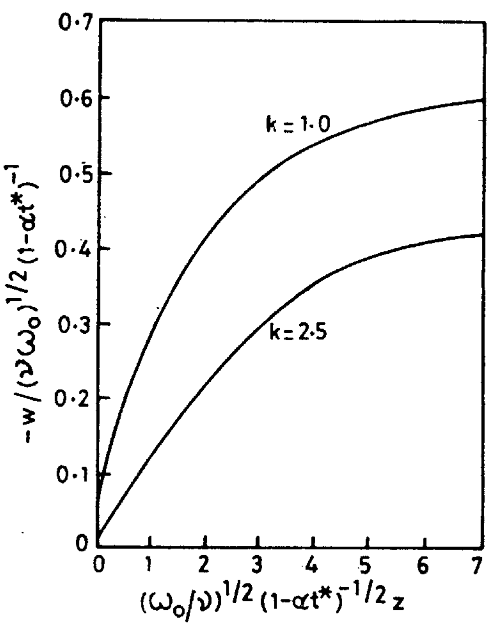

FIG. 3. Dependence of fluid velocities in axial direction upon particle loading (β = 0.5 and α = 1.0).

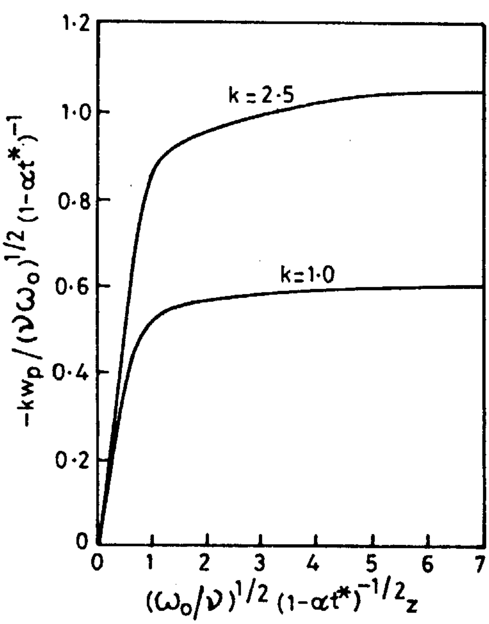

FIG. 4. Dependence of particle velocities in axial direction upon particle loading (β = 0.5 and α = 1.0).

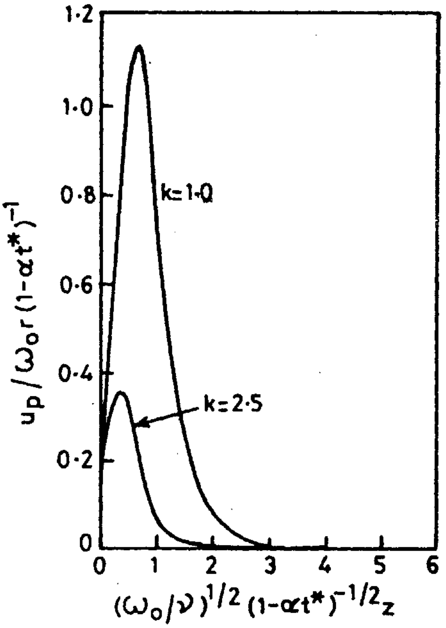

FIG. 5. Dependence of particle velocities in radial direction upon particle loading (β = 0.5 and α = 1.0).

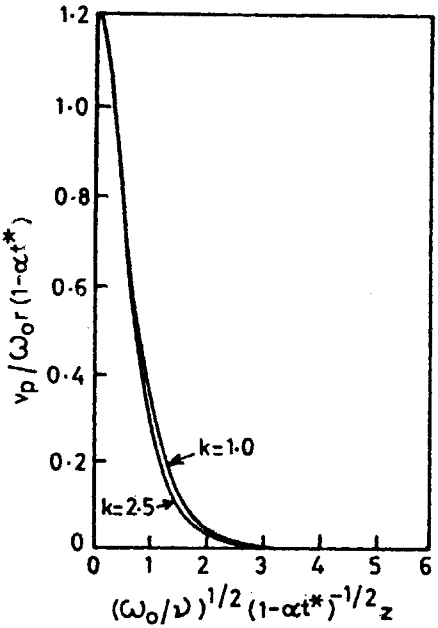

FIG. 6. Dependence of particle velocities in tangential direction upon particle loading (β = 0.5 and α = 1.0).

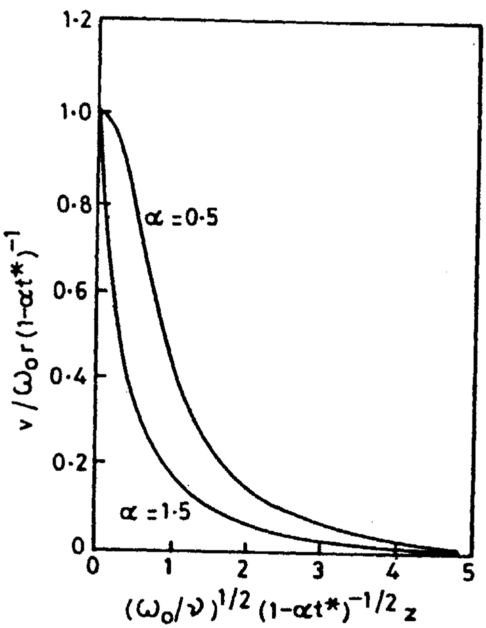

FIG. 7. Dependence of particle velocities in radial direction variation α (β = 0.5 and α = 1.0).

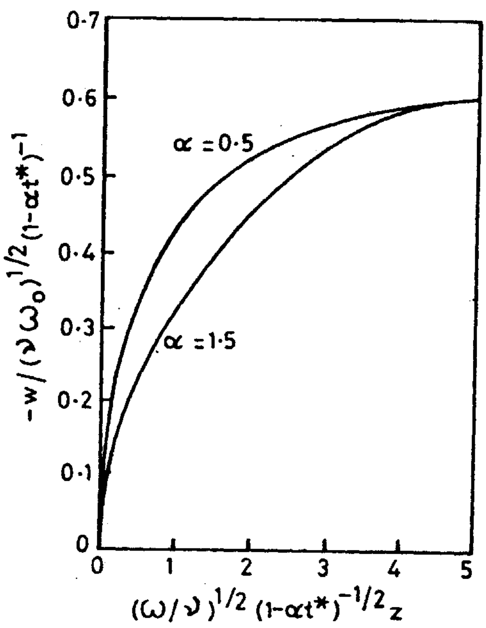

FIG. 8. Dependence of fluid velocities in axial direction variation α (β = 0.5 and α = 1.0).

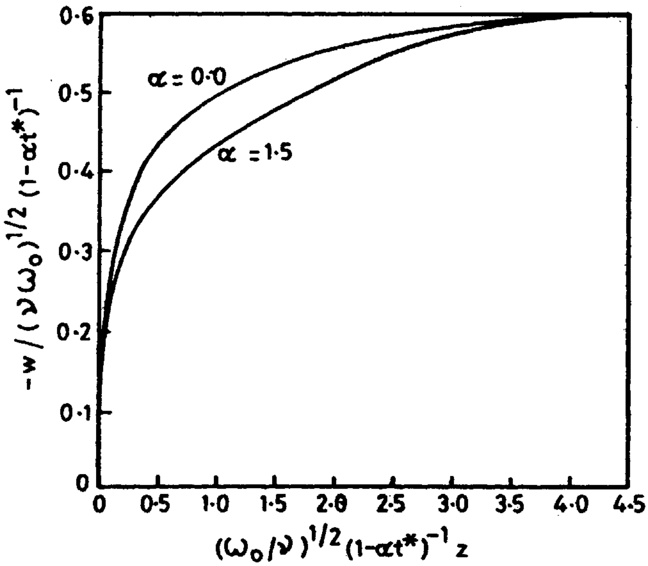

FIG. 9. Dependence of particle velocities in axial direction upon particle loading ($\beta = 0.5$ and $\alpha = 0.0$).

RESULTS AND DISCUSSION

The fluid velocity profiles are shown in Figs. 1-3 for various particle loading ($k = 1$ and $k = 2.5$) and in Figs. 7 and 8 for various value of parameter α. The flow is like Prandtl's boundary layer type, the boundary layer thickness for both fluid and particle cloud are approximately the same. It is observed that radial and tangential components of fluid velocity are becoming zero for large values of η, at the same time the axial component of fluid velocity attains a constant value. The result is similar to those obtained by Zung[13] for the larger value of η and $\alpha = 0$ (Fig. 9). The corresponding particle velocity components are shown in Figs. 4-6 the radial component of velocity decreases as k increases. The tangential component of particle velocity is maximum on the disk and it decreases and tends to zero approximately for $\eta = 4$, similar behaviour is shown by radial component of velocity. The values of u_p very close to disk are depending upon the particle loading k. The variation of particle density and fluid pressure is also studied for various particle loading k and parameter α, it is found that $\rho_p/\rho k$ tends to 1 as η tends to ∞. It is worth noting that $\rho_p/\rho k$ for $\alpha = 0$ tends to 1 for very small value of η, whereas for $\alpha > 0$ and $\alpha < 0$ it tends to 1 for approximately $\eta \doteq 4$. The pressure attains its maximum on the disk. It is found that numerical integration is possible for larger values of η for $\alpha > 0$ as compared to the case of $\alpha = 0$ and $\alpha < 0$. The particle and fluid velocity component showing the similar nature, but decreases as α increases for both accelerating ($\alpha > 0$) or decelerating ($\alpha < 0$) flow. We found that maximum value of F and F_p are more for decelerating ($\alpha < 0$) than for accelerating flow. It may be noted that result for the case $\alpha = 0$ are in full agreement with Zung[13].

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