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GENERALIZED LOCALLY CLOSED SETS AND *GLC*-CONTINUOUS FUNCTIONS

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In this paper we introduce generalized locally closed sets and different notions of generalizations of continuous maps in a topological space and discuss some of their properties.

1. INTRODUCTION

According to Bourbaki[1] a subset of a topological space is locally closed if it is the intersection of an open set and a closed set. Stone[6] has used the term *FG* for a locally closed subset. By using the concept of a locally closed set Ganster and Reilly[3] introduced *LC*-irresoluteness, *LC*-continuity and sub-*LC*-continuity and discussed some properties of these functions. We introduce and investigate the concept of "generalized locally closed set" in section 2 and the classes of *GLC*-irresolute maps and *GLC*-continuous maps in section 3.

2. GENERALIZED LOCALLY CLOSED SETS

Throughout this paper, (X, τ) denotes a topological space with a topology τ on which no separation axioms are assumed unless explicitly stated and, for a subset A of X, $Cl\ (A)$ and $Int\ (A)$ denote the closure of A and the interior of A with respect to (X, τ) respectively. Let $P(X)$ be the power set of X. Before entering into our work we recall the following definitions.

Definition 2.1 — A subset S of (X, τ) is called g-closed[4] if $Cl\ (S) \subset G$ whenever $S \subset G$ and G is open in (X, τ). A subset S of (X, τ) is called g-open if its complement $X - S$ is g-closed.

Remark 2.2[4] : It has been proved that closed set implies g-closed set, g-closed set need not imply closed set, open set implies g-open set and g-open set need not imply open set.

Definition 2.3 — A subset S of (X, τ) is called locally closed[1] if $S = G \cap F$ where $G \in \tau$ and F is closed in (X, τ).

Remark 2.4 : The following are well known.

(i) A subset S of (X, τ) is locally closed if and only if its complement $X - S$ is the union of an open set and a closed set.

(ii) Every open (resp. closed) subset of X is locally closed.

(iii) The complement of a locally closed set need not be locally closed.

Now we introduce the following.

Definition 2.5 — A subset S of (X, τ) is called generalized locally closed set (briefly, *glc*) if $S = G \cap F$ where G is g-open in (X, τ) and F is g-closed in (X, τ). Every g-closed set (resp. g-open set) is *glc*.

The collection of all generalized locally closed sets (resp. locally closed sets) of (X, τ) will be denoted by

$GLC(X, \tau)$ (resp. $LC(X, \tau)$ (Bourbaki[1], p.19).

The following two collections of subsets of (X, τ), i.e. $GLC^*(X, \tau)$ and $GLC^{**}(X, \tau)$, are defined as follows :

Definition 2.6 — For a subset S of (X, τ), $S \in GLC^*$ (X, τ) if there exist a g-open set G and a closed set F of (X, τ), respectively, such that $S = G \cap F$.

Definition 2.7 — For a subset S of (X, τ), $S \in GLC^{**}$ (X, τ) if there exist an open set G and a g-closed set F of (X, τ), respectively, such that $S = G \cap F$.

Proposition 2.8 — Let S be a subset of (X, τ).

(i) If S is locally closed, then $S \in GLC^*$ (X, τ) and $S \in GLC^{**}$ (X, τ), however not conversely.

(ii) If $S \in GLC^*$ (X, τ) or $S \in GLC^{**}$ (X, τ), then S is *glc*.

PROOF : The proofs are obvious from Remark 2.2, definitions and the following Example :

Example 2.9 — Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. Then $LC(X, \tau) = \{\phi, \{a\}, \{b, c\}, X\}$ and $GLC^*(X, \tau) = GLC^{**}$ $(X, \tau) = GLC(X, \tau) = P(X)$ because ϕ, X, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ are the g-closed sets of (X, τ). Then, LC (X, τ) is a proper subset of $GLC(X, \tau)$.

The following result is a characterization of $GLC^*(X, \tau)$ (cf. Ganster and Reilly[3], Proposition 1).

Theorem 2.10 — For a subset S of (X, τ), the following are equivalent :

(i) $S \in GLC^*$ (X, τ).

(ii) $S = P \cap Cl(S)$ for some g-open set P.

(iii) $Cl(S) - S$ is g-closed.

(iv) $S \bigcup (X - Cl(S))$ is g-open.

PROOF : (i) ⇒ (ii) There exist a g-open set P and a closed set F such that $S = P \bigcap F$. Since $S \subset P$ and $S \subset Cl(S)$, $S \subset P \bigcap Cl(S)$. Conversely, since $Cl(S) \subset F$ we have $S = P \bigcap F \supset P \bigcap Cl(S)$. Therefore we have $S = P \bigcap Cl(S)$.

(ii) ⇒ (i) Since P is g-open and $Cl(S)$ is closed, $P \bigcap Cl(S) \in GLC^*$ (X, τ) by Definition 2.6.

(ii) ⇒ (iii) : It follows from assumption and Corollary 2.7 of Levine[4] that $Cl(S) - S = Cl(S) \bigcap (X - P)$ is g-closed.

(iii) ⇒ (ii) : Let $U = X - (Cl(S) - S)$. By assumption, U is g-open and $S = U \bigcap Cl(S)$ holds.

(iii) ⇒ (iv) : Let $F = Cl(S) - S$. Then, $S \bigcup (X - Cl(S))$ is g-open, since $X - F = S \bigcup (X - Cl(S))$ holds and $X - F$ is g-open.

(iv) ⇒ (iii) Let $U = S \bigcup (X - Cl(S))$. Then, $X - U$ is g-closed and $X - U = Cl(S) - S$ holds. It completes the proof.

Remark 2.11 : As can be seen from Example 2.9 above, it is not true that $S \in GLC^*$ (X, τ) if and only if $S \subset Int(S \bigcup (X - Cl(S)))$ (cf. Ganster and Reilly[3] Proposition 1(v)).

In fact, let $S = \{a, b\}$ in the topological space (X, τ) of Example 2.9. Then, $\text{Int}(S \bigcup (X - Cl(S))) = Int(\{a, b\}) = \{a\} \not\supset S$ and $S \in GLC^*$ (X, τ).

We need the following definition to get a corollary to this theorem (cf. Ganster and Reilly[3], Corollary 1(v)).

Definition 2.12 — A topological space (X, τ) is called g-submaximal if every dense subset is g-open.

Corollary 2.13 — A topological space (X, τ) is g-submaximal if and only if $P(X) = GLC^*(X, \tau)$ holds.

PROOF : *Necessity* — Let $S \in P(X)$ and let $U = S \bigcup (X - Cl(S))$. Then, it is easily verified that $X = Cl(U)$, i.e. U is a dense subset of (X, τ). By assumption, U is g-open. Therefore, it follows from Theorem 2.10 that $S \in GLC^*$ (X, τ), and hence $P(X) = GLC^*(X, \tau)$ holds.

Sufficiency — Let S be a dense subset of (X, τ). Then, it follows from assumptions and Theorem 2.10(iv) that $S \bigcup (X - Cl(S)) = S$ holds, $S \in GLC^*$ (X, τ) and S is g-open. This implies (X, τ) is g-submaximal.

Remark 2.14 : It follows from definitions that if (X, τ) is submaximal then it is g-submaximal. As can be seen from Example 2.9 above, its converse is not true. The topological space (X, τ) of Example 2.9 is g-submaximal since $P(X) = GLC^*$

(X, τ) holds. However, (X, τ) is not submaximal since $LC(X, \tau) \neq P(X)$, (cf. Ganster and Reilly[3], Corollary 1).

Proposition 2.15 — For a subset S of (X, τ), if $S \in GLC^{**}$ (X, τ) then there exists an open set P such that $S = P \cap Cl^{*}(S)$ (here $Cl^{*}(S)$ is the closure of S defined by Dunham[2]).

PROOF : There exist an open set P and a g-closed set F such that $S = P \cap F$. Since $S \subset P$ and $S \subset Cl^{*}(S)$ we have $S \subset P \cap Cl^{*}(S)$. Conversely, since $Cl^{*}(S) \subset F\ (= Cl^{*}(F))$ holds we have $S = P \cap F \supset P \cap Cl^{*}(S)$ and hence $S = P \cap Cl^{*}(S)$.

The following results are basic properties of "generalized locally closed sets" (cf. Ganster and Reilly[3], Propositions 3, 4, 5 and Theorem 1).

Proposition 2.16 — Let A and B be subsets of (X, τ).

(i) If $A \in GLC^{*}$ (X, τ) and $B \in GLC^{*}$ (X, τ) then $A \cap B \in GLC^{*}$ (X, τ).

(ii) If $A \in GLC^{**}$ (X, τ) and B is closed or open then $A \cap B \in GLC^{**}$ (X, τ).

(iii) If $A \in GLC$ (X, τ) and B is g-open or closed then $A \cap B \in GLC(X, \tau)$.

PROOF : (i) It follows from Theorem 2.10 (ii) that there exist g-open sets P and Q such that $A = P \cap Cl(A)$ and $B = Q \cap Cl(B)$. Then, $A \cap B \in GLC^{*}$ (X, τ) since $P \cap Q$ is g-open by Theorem 2.4 of Levine[4] and $Cl(A) \cap Cl(B)$ is closed.

(ii) It follows from Definition 2.7 that there exist an open set G and g-closed set F such that $A \cap B = G \cap F \cap B$. First suppose that B is open. Then, it is shown that $A \cap B \in GLC^{**}$ (X, τ). Next suppose that B is closed. By using Corollary 2.7 of Levine[4] it is proved that $F \cap B$ is g-closed and so $A \cap B \in GLC^{**}$ (X, τ).

(iii) It follows from Definition 2.5 that there exist a g-open set G and g-closed set F such that $A \cap B = G \cap F \cap B$. First suppose that B is g-open. Then, by using Theorem 2.4 of Levine[4], it is shown that $A \cap B \in GLC^{**}$ (X, τ). Next suppose that B is closed. By using Corollary 2.7 of Levine[4], it is proved that $F \cap B$ is g-closed and so $A \cap B \in GLC\,(X, \tau)$.

Proposition 2.17 — Let A and Z be subsets of (X, τ) and let $A \subset Z$.

(i) If Z is g-open in (X, τ) and $A \in GLC^{*}$ $(Z, \tau \mid Z)$, then $A \in GLC^{*}$ (X, τ).

(ii) If Z is g-closed in (X, τ) and $A \in GLC^{**}$ $(Z, \tau \mid Z)$, then $A \in GLC^{**}$ (X, τ).

(iii) If Z is g-closed and g-open in (X, τ) and $A \in GLC(Z, \tau \mid Z)$, then $A \in GLC$ (X, τ).

PROOF : (i) It follows from Theorem 2.10 that there exists a g-open set G of $(Z, \tau \mid Z)$ such that $A = G \bigcap Cl_Z(A)$, where $Cl_Z(A) = Z \bigcap Cl(A)$. By using Theorem 4.6 of Levine[4] and Definition 2.6, it is proved that $A = (Z \bigcap G) \bigcap Cl(A) \in GLC^*(X, \tau)$.

(ii) There exists an open set G of $(Z, \tau \mid Z)$ and g-closed set F of $(Z, \tau \mid Z)$ such that $A = G \bigcap F$. By Theorem 2.6 of Levine[4] F is g-closed in (X, τ). Since $G = B \bigcap Z$ for some open set B of (X, τ), $A = (Z \bigcap B) \bigcap F = F \bigcap B$ holds we have $A \in GLC^{**}(X, \tau)$.

(iii) There exist a g-open set G of $(Z, \tau \mid Z)$ and a g-closed set F of $(Z, \tau \mid Z)$ such that $A = G \bigcap F$. By using Theorems 2.6 and 4.6 of Levine[4], it is proved that $A \in GLC(X, \tau)$.

Remark 2.18 : The following example shows that one of the assumptions of Proposition 2.17(i), i.e. Z is g-open, cannot be removed.

Example 2.19 — Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Let $\mathcal{V}$ denote the collection of all g-open sets of (X, τ).

Then we have $\mathcal{V} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Put $Z = A = \{a, c\}$. It is shown that Z is not g-open and $A \in GLC^*(Z, \tau \mid Z)$.

However, $A \notin GLC^*(X, \tau)$ since $GLC^*(X, \tau) = P(X) - \{\{a, c\}\}$.

Proposition 2.20 — Suppose that the collection of all g-open sets of (X, τ) is closed under finite unions. Let $A \in GLC^*(X, \tau)$ and $B \in GLC^*(X, \tau)$. If A and B are separated. i.e. $A \bigcap Cl(B) = \phi$ and $B \bigcap Cl(A) = \phi$, then $A \bigcup B \in GLC^*(X, \tau)$.

PROOF : By using Theorem 2.10 there exists g-open sets G and S of (X, τ) such that $A = G \bigcap Cl(A)$ and $B = S \bigcap Cl(B)$. Put $U = G \bigcap (X - Cl(B))$ and $V = S \bigcap (X - Cl(A))$. Then, $A = U \bigcap Cl(A)$ and $B = V \bigcap Cl(B)$ hold, and $U \bigcap Cl(B) = \phi$ and $V \bigcap Cl(A) = \phi$ hold. It follows from Theroem 2.4 of Levine[4] that U and V are g-open sets of (X, τ). Therefore, since $A \bigcup B = (U \bigcup V) \bigcap (Cl(A \bigcup B))$ and $U \bigcup V$ is g-open by assumption, we have $A \bigcup B \in GLC^*(X, \tau)$.

Remark 2.21 : Example 2.19 shows that one of assumptions of Proposition 2.20, i.e. A and B are separated, cannot be removed. $\mathcal{V}$ is closed under finite unions, and $\{a\} \in GLC^*(X, \tau)$ and $\{c\} \in GLC^*(X, \tau)$. However, $\{a\}$ and $\{c\}$ are not separated and $\{a, c\} \notin GLC^*(X, \tau)$.

Proposition 2.22 — Let $\{Z_i \mid i \in \Lambda\}$ be a finite g-closed cover of (X, τ), i.e. $X = \bigcup \{Z_i \mid i \in \Lambda\}$, and let A be a subset of (X, τ). If $A \bigcap Z_i \in GLC^{**}(Z_i, \tau \mid Z_i)$ for each $i \in \Lambda$, then $A \in GLC^{**}(X, \tau)$.

PROOF : For each $i \in \Lambda$ there exist an open set $U_i \in \tau$ and a g-closed set F_i of $(Z_i, \tau | Z_i)$ such that $A \cap Z_i = U_i \cap (Z_i \cap F_i)$. Then, $A = \bigcup \{A \cap Z_i \,|\, i \in \Lambda\} = [\bigcup \{U_i \,|\, i \in \Lambda\}] \cap [\bigcup \{Z_i \cap F_i \,|\, i \in \Lambda\}]$, and hence $A \in GLC^{**}(X, \tau)$ by Theorems 2.6 and 2.4 of Levine[4].

Proposition 2.23 — Let (X, τ) and (Y, σ) be topological spaces.

(i) If $A \in GLC(X, \tau)$ and $B \in GLC(Y, \sigma)$, then $A \times B \in GLC(X \times Y, \tau \times \sigma)$.

(ii) If $A \in GLC^*(X, \tau)$ and $B \in GLC^*(Y, \sigma)$, then $A \times B \in GLC^*(X \times Y, \tau \times \sigma)$.

(iii) If $A \in GLC^{**}(X, \tau)$ and $B \in GLC^{**}(Y, \sigma)$, then $A \times B \in GLC^{**}(X \times Y, \tau \times \sigma)$.

PROOF : (i) There exist g-open sets G and G' of (X, τ) and (Y, σ), respectively, and g-closed sets S and S' of (X, τ) and (Y, σ), respectively, such that $A = G \cap S$ and $B = G' \cap S'$. Then, $A \times B = (G \times G') \cap (S \times S')$ holds and hence $A \times B \in GLC(X \times Y, \tau \times \sigma)$ by Theorems 7.1 and 7.3 of Levine[4].

(ii) & (iii) The proofs are similar to (i).

3. GLC-FUNCTIONS AND SOME OF THEIR PROPERTIES

Let $f : (X, \tau) \to (Y, \sigma)$ be a function between topological spaces (X, τ) and (Y, σ). Ganster and Reilly[3] defined three distinct notions of LC-continuity, i.e. LC-irresoluteness, LC-continuity and sub-LC-continuity. In this section we define generalizations of LC-irresolute functions, LC-continuous functions and sub-LC-continuous functions and study some of their properties.

Definition 3.1 — A function $f : (X, \tau) \to (Y, \sigma)$ is called GLC-irresolute (resp. GLC^*-irresolute, resp. GLC^{**}-irresolute) if $f^{-1}(V) \in GLC(X, \tau)$ (resp. $f^{-1}(V) \in GLC^*(X, \tau)$, resp. $f^{-1}(V) \in GLC^{**}(X, \tau)$) for each $V \in GLC(Y, \sigma)$ (resp. $V \in GLC^*(Y, \sigma)$, resp. $V \in GLC^{**}(Y, \sigma)$).

Definition 3.2 — A function $f : (X, \tau) \to (Y, \sigma)$ is called GLC-continuous (resp. GLC^*-continuous, resp. GLC^{**}-continuous) if $f^{-1}(V) \in GLC(X, \tau)$ (resp. $f^{-1}(V) \in GLC^*(X, \tau)$, resp. $f^{-1}(V) \in GLC^{**}(X, \tau)$) for each $V \in \sigma$.

Proposition 3.3 — Let $f : (X, \tau) \to (Y, \sigma)$ be a function.

(i) If f is LC-continuous, then it is GLC^*-continuous and GLC^{**}-continuous.

(ii) If f is GLC^*-continuous or GLC^{**}-continuous, then it is GLC-continuous.

(iii) If f is GLC-irresolute (resp. GLC^*-irresolute, resp. GLC^{**}-irresolute), then it is GLC-continuous (resp. GLC^*-continuous, resp. GLC^{**}-continuous).

(iv) If f is continuous and closed, then f is GLC^*-irresolute, GLC^{**}-irresolute and GLC-irresolute.

PROOF : (i) Suppose that f is LC-continuous. Let V be an open set of (Y, σ). Then $f^{-1}(V)$ is locally closed in (X, τ) by definition. By Proposition 2.8 it is obtained that f is GLC^*-continuous and GLC^{**}-continuous.

(ii) & (iii) The proofs are obvious from definitions.

(iv) It is proved by Theorem 6.3 of Levine[4] and definitions.

Remark 3.4 : Converses of Proposition 3.3 need not be true as seen from the following examples.

Example 3.5 — Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = P(Y)$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity. Since $GLC^*(X, \tau) = GLC^{**}(X, \tau) = GLC(X, \tau) = P(X)$, $LC(X, \tau) = \{\phi, \{a\}, \{b, c\}, X\}$ and $LC(Y, \sigma) = GLC(Y, \sigma) = GLC^*(Y, \sigma) = GLC^{**}(Y, \sigma) = P(Y)$, f is not LC-continuous; it is GLC^*-continuous, GLC^{**}-continuous and GLC^*-irresolute.

Example 3.6 — Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = f(c) = a$ and $f(b) = b$. Then, $GLC^{**}(X, \tau) = GLC(X, \tau) = P(X)$, $LC(X, \tau) = GLC^*(X, \tau) = P(X) - \{\{a, c\}\}$ and $GLC(Y, \sigma) = GLC^{**}(Y, \sigma) = GLC^*(Y, \sigma) = P(Y)$.

Therefore, f is not GLC^*-continuous; but it is GLC-continuous.

Example 3.7 — The function of Example 3.6 is not GLC^*-continuous; but it is GLC^{**}-continuous.

Example 3.8 — Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then, f is not GLC^*-irresolute; but it is LC-continuous.

From Proposition 3.3 and Examples 3.4-3.8, we have the following diagram :

$$\begin{tikzcd}
 & GLC^*\text{-irresoluteness} \arrow[dl, shift left, "/" marking] \arrow[d, shift right] \arrow[dr, shift left] & \\
LC\text{-continuity} \arrow[ur, shift left, "/" marking] \arrow[r, shift left] \arrow[dr, shift left] & GLC^*\text{-continuity} \arrow[l, shift left, "/" marking] \arrow[u, shift right, "/" marking] \arrow[r, shift left] & GLC\text{-continuity} \arrow[l, shift left, "/" marking] \arrow[ul, shift left, "/" marking] \\
 & GLC^{**}\text{-continuity} \arrow[ul, shift left, "/" marking] \arrow[u, "/" marking] \arrow[ur] &
\end{tikzcd}$$

where $A \to B$ (resp. $A \not\to B$) represents that A implies B (resp. A does not always imply B).

The following result is an immediate consequence of Corollary 2.13 (cf. Ganster and Reilly[3], Proposition 6).

Proposition 3.9 — A topological space (X, τ) is g-submaximal if and only if every function having (X, τ) as its domain is GLC^*-continuous.

PROOF : *Necessity* — Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a function. By Corollary 2.13 we have that $f^{-1}(V) \in GLC^*(X, \tau) = P(X)$ for each open set V of (Y, σ). Therefore, f is GLC^*-continuous.

Sufficiency — Let $Y = \{0, 1\}$ be the Sierpinski space[5] with topology $\sigma = \{\phi, \{0\}, Y\}$. Let V be a subset of (X, τ) and $f : (X, \tau) \to (Y, \sigma)$ a function defned by $f(x) = 0$ for every $x \in V$ and $f(x) = 1$ for every $x \notin V$. It follows from assumption that f is GLC^*-continuous and hence $f^{-1}(\{0\}) = V \in GLC^*(X, \tau)$. Therefore we have $P(X) = GLC^*(X, \tau)$ and so (X, τ) is g-submaximal by Corollary 2.13.

Proposition 3.10 — If $f : (X, \tau) \to (Y, \sigma)$ is GLC^{**}-continuous and a subset B is closed in (X, τ), then the restriction of f to B, say $f \mid B : (B, \tau \mid B) \to (Y, \sigma)$ is GLC^{**}-continuous.

PROOF : Let V be an open set of (Y, σ). Then, $f^{-1}(V) = G \bigcap F$ for some open set $G \in \tau$ and g-closed set F of (X, τ). By using Corollary 2.7 and Theorem 2.9 of Levine[4], we have that $(f|B)^{-1}(V) = (G \bigcap B) \bigcap (F \bigcap B) \in GLC^{**}(B, \tau|B)$. This implies that $f \mid B$ is GLC^{**}-continuous.

We recall the definition of the combination of two functions. Let $X = A \bigcup B$ and $f : A \to Y$ and $h : B \to Y$ be two functions. We say that f and h are compatible if $f|A \bigcap B = h|A \bigcap B$. Then, we can define a function $f \nabla h : X \to Y$ as follows :

$(f \nabla h)(x) = f(x)$ for every $x \in A$ and $(f \nabla h)(x) = h(x)$ for every $x \in B$. The function $f \nabla h : X \to Y$ is called the combination of f and h.

Theorem 3.11 — Let $X = A \bigcup B$, where A and B are g-closed sets of (X, τ), and $f: (A, \tau|A) \to (Y, \sigma)$ and $h : (B, \tau \mid B) \to (Y, \sigma)$ be compatible functions. If f and h are GLC^{**}-continuous (resp. GLC^{**}-irresolute), then $f \nabla h : (X, \tau) \to (Y, \sigma)$ is GLC^{**}-continuous (resp. GLC^{**}-irresolute).

PROOF : Let $V \in \sigma$ (resp. $V \in GLC^{**}(Y, \sigma)$). Then, $(f \nabla h)^{-1}(V) \bigcap A = f^{-1}(V)$ and $(f \nabla h)^{-1}(V) \bigcap B = h^{-1}(V)$ hold. By assumptions we have $(f \nabla h)^{-1}(V) \bigcap A \in GLC^{**}(A, \tau|A)$ and $(f \nabla h)^{-1}(V) \bigcap B \in GLC^{**}(B, \tau|B)$. Therefore, it follows from Proposition 2.22 that $(f \nabla h)^{-1}(V) \in GLC^{**}(X, \tau)$ and hence $f \nabla h$ is GLC^{**}-continuous (resp. GLC^{**}-irresolute).

Remark 3.12 : Example 3.6 shows that the pasting lemma for GLC^{*}-continuous functions is not true. Let $(X, \tau), (Y, \sigma)$ and $f : (X, \tau) \to (Y, \sigma)$ be the topological spaces and the function in Example 3.6. Let $A = \{a, c\}$ and $B = \{b, c\}$. Then $\{A, B\}$ is a g-closed cover of X, and $f \mid A : (A, \tau|A) \to (Y, \sigma)$ and $g \mid B : (B, \tau|B) \to (Y, \sigma)$ are GLC^{*}-continuous functions. However, the combination $(f|A) \nabla (g|B) = f$ is not GLC^{*}-continuous.

Concerning compositions of functions we have the following :

(a) The composition of two GLC-irresolute (resp. GLC^{*}-irresolute, resp. GLC^{**}-irresolute) functions is clearly GLC-irresolute (resp. GLC^{*}-irresolute, resp. GLC^{**}-irresolute).

(b) The composition $g \circ f$ of a GLC-continuous (resp. GLC^{*}-continuous, resp. GLC^{**}-continuous) function f and a continuous function g is clearly GLC-continuous (resp. GLC^{*}-continuous, resp. GLC^{**}-continuous).

In the end of this section we generalize the notion of sub-LC-continuous functions (cf. Ganster and Reilly[3], Propositions 11 and 12).

Definition 3.13 — A function $f : (X, \tau) \to (Y, \sigma)$ is called sub-GLC^{*}-continuous if there is a basis $\mathcal{B}$ for (Y, σ) such that $f^{-1}(U) \in GLC^{*}(X, \tau)$ for each $U \in \mathcal{B}$.

Proposition 3.14 — Let $f : (X, \tau) \to (Y, \sigma)$ be a function.

(i) f is sub-GLC^*-continuous if and only if there is a subbasis C for (Y, σ) such that $f^{-1}(U) \in GLC^*(X, \tau)$ for each $U \in C$.

(ii) If f is sub-LC-continuous then f is sub-GLC^*-continuous.

PROOF : (i) *Necessity* — It follows from assumption that there is a basis $\mathcal{B}$ for (Y, σ) such that $f^{-1}(U) \in GLC^*(X, \tau)$ for each $U \in \mathcal{B}$. Since $\mathcal{B}$ is also a subbasis for (Y, σ), the proof is obvious.

Sufficiency — For a subbasis C, let $C_\delta = \{A \subset Y \mid A$ is an intersection of finitely many sets belonging to $C\}$. Then, C_δ is a basis for (Y, σ). For a $U \in C_\delta$, $U = \bigcap \{F_i \mid F_i \in C,\ i \in \Lambda\}$ where Λ is a finite set. By using Proposition 2.16(i) and assumption we have $f^{-1}(U) = \bigcap \{f^{-1}(F_i) \mid i \in \Lambda\} \in GLC^*(X, \tau)$.

(ii) It is obtained by (i) and Definition (iii) of Ganster and Reilly[3].

Remark 3.15 : The following example shows that the converse of Proposition 3.14(ii) is not always true.

Example 3.16 — Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Since $LC(X, \tau) = \{\phi, \{a\}, \{b, c\}, X\}$, $GLC^*(X, \tau) = P(X)$ and a family $\mathcal{B} = \{\{a, b\}, Y\}$ is a base for (Y, σ). f is not sub-LC-continuous; it is sub-GLC^*-continuous.

Example 3.17 — The function f of Example 3 of Ganster and Reilly[3] (p. 423) is an example of sub LC-continuous. By Proposition 3.14 (ii), it is also sub-GLC^*-continuous. However, f is not GLC^*-continuous. In fact, similarly as in Ganster and Reilly[3], let $U = R - (\{0\} \cup \{1/n \mid n \in N, n \geq 2\})$. Then, U is open in Y, $f^{-1}(U) = U \cup \{0\}$ and $Cl(f^{-1}(U)) - f^{-1}(U) = \{1/n \mid n \in N, n \geq 2\}$ is not g-closed. By using Theorem 2.10, $f^{-1}(U) \in GLC^*(X, \tau)$. Hence f is not GLC^*-continuous. Moreover, it is not LC-continuous[3].

From Proposition 3.14(ii), Remark 3.15, Example 3.17, Definition 3.13, Proposition 3.3(i), Example 3.5, and Definition (iii) of Ganster and Reilly[3], we have the following diagram :

$$\begin{tikzcd}
\text{sub-}LC\text{-continuity} \arrow[r, shift left] & \text{sub-}GLC^*\text{-continuity} \arrow[l, shift left, "\mid" marking] \arrow[d, shift left, "\mid" marking] \\
LC\text{-continuity} \arrow[r, shift left] \arrow[u, shift left] & GLC^*\text{-continuity} \arrow[l, shift left, "\mid" marking] \arrow[u, shift left]
\end{tikzcd}$$

where $A \to B$ (resp. $A \nrightarrow B$) represents that A implies B (resp. A does not always imply B).

Proposition 3.20 — (i) If $f : (X, \tau) \to (Y, \sigma)$ and $h : (X', \tau') \to (Y', \sigma')$ are sub-GLC^*-continuous, then $f \times h : (X \times X', \tau \times \tau') \to (Y \times Y', \sigma \times \sigma')$ is sub-GLC^*-continuous.

(ii) If $f : (X, \tau) \to (Y, \sigma)$ is sub-GLC^*-continuous, then $(1 \times f)\,\Delta : (X, \tau) \to (X \times Y, \tau \times \sigma)$ is sub-GLC^*-continuous, where $1 : (X, \tau) \to (X, \tau)$ is the identity and $\Delta : (X, \tau) \to (X \times X, \tau \times \tau)$ is the diagonal map defined by $\Delta(x) = (x, x)$ for every $x \in X$.

PROOF : (i) It follows from assumptions that there exist a basis $\mathcal{B}$ for (Y, σ) and a basis $\mathcal{B}'$ for (Y', σ') satisfying the condition of Definition 3.13 respectively. Then, $\mathcal{B}'' = \{U \times V \mid U \in \mathcal{B}, V \in \mathcal{B}'\}$ is a basis for the product space $(X \times X', \tau \times \tau')$. It follows from Proposition 2.23 that $(f \times h)^{-1}(U \times V) = f^{-1}(U) \times h^{-1}(V) \in GLC^*(X \times X', \tau \times \tau')$ for every $U \times V \in \mathcal{B}''$. Therefore, $f \times h$ is sub-GLC^*-continuous.

(ii) The proof is similar to (i) using Proposition 2.16.

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