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ON BITOPOLOGICAL *QHC*-SPACES

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The present paper is a continuation of the study of the well known concept of bitopological quasi *H*-closed spaces from altogether new standpoints. Some interesting properties of such spaces are achieved via different approaches viz. covers, nets with well ordered sets as domains, θ-complete adherent points of sets, partial order relation and a few types of functions and multifunctions. These properties turn out to be new characterizations of almost regular bitopological spaces to be quasi *H*-closed. An application in the form of a fixed set theorem for a sort of multifunctions is also obtained for a bitopological quasi *H*-closed space.

1. INTRODUCTION AND PRELIMINARIES

H-closed and quasi *H*-closed (*QHC*, for short) topological spaces are considered to be interesting and important topics of study for as long as the last seventy years or so. The intensive study of such spaces by eminent topologists during this long period has motivated many others to generalize the results to bitopology and even to fuzzy setting.

The investigation of bitopological *QHC*-spaces was initiated with the paper of Mukherjee[3], which was followed by their further study by Kariofillis[2] and Mukherjee and Coworkers[4, 6, 7]. The intent of this article is to pursue the above trend, wherein we enunciate some more results concerning the bitopological *QHC*-spaces, specially in the presence of biotopological almost regularity on the underlying spaces.

In the characterizations of bitopological *QHC*-spaces viz. *ij*-*QHC* and pairwise *QHC*-spaces, the basic appliances have always been the filterbases, nets and covers. In section 2, we carry on with the process of characterizations with the same set of tools, but with new approaches and orientations. We have introduced the notion of *ij*-θ-complete adherent point of sets, and use nets with well-ordered sets as domain to derive results analogous to those for a compact topological space. We find that for *ij*-almost regular spaces, a type of sets viz. *ij*-θ-open sets play the same vital role vis-a-vis *ij*-quasi *H*-closedness as that done by open sets towards compactness. At the end of the section, we establish, as an interesting application, a 'fixed set theorem' for a type of multifunction on an *ij*-*QHC*-space.

The third section envisages, to begin with, the introduction and characterizations of two types of multifunctions. A sort of compatibility condition for partial order relations on a set along with two kinds of functions between bitopological spaces are then defined. All these are done only to pave the way for the subsequent study of *ij-QHC*-spaces. Certain properties of *ij-QHC*-spaces are then established, the same being found to be necessary and sufficient conditions for *ij*-almost regular spaces to be *ij-QHC*. In the literature, it is found that Urysohn, *QHC* topological spaces are of special interest to the topologists. Since a bitopological Urysohn, *QHC*-space is known[6] to be *ij*-almost regular, it turns out that our results are equally valid in the former setting too.

Unless otherwise is stated, by a space X (or Y) we shall mean a bitopological space (X, Q_1, Q_2) (resp. (Y, P_1, P_2)). For brevity, the word 'neighbourhood' will be written as nbd. The notation $f : X \to Y$ will mean that f is a function from the space (X, Q_1, Q_2) to the space (Y, P_1, P_2), whereas by a multifunction $F : X \to Y$ is meant, as usual, a function from X to $\mathcal{P}(Y)\backslash\{\phi\}$, where $\mathcal{P}(Y)$ denotes the power set of Y. To avoid repetitions, we shall all through adhere to the following conventions regarding the use of i and j : in any context of discussion, whenever i and j both will occur, it is to be understood that $i, j \in \{1, 2\}$ and $i \neq j$. In the case when only i (or j) appears, it will mean that $i = 1, 2$ (resp. $j = 1, 2$). For a subset A of a space (X, Q_1, Q_2), the interior and closure of A in (X, Q_i) will be denoted by Q_i-*int* A and Q_i-*cl* A respectively, where according to our convention $i = 1, 2$. Also we shall, in general, use '*ij*' in place of $Q_i Q_j$ and $P_i P_j$ where there is no scope for confusion.

A point x in a space (X, Q_1, Q_2), is called an *ij*-θ-adherent point[1, 2] of a subset A of X if for any Q_i-open nbd U of x, $A \cap Q_j\text{-}cl\ U \neq \phi$. The set of all *ij*-$\theta$-adherent points of A is called the *ij*-θ-closure of A and is denoted by *ij*-θ-*cl* A (Mukherjee *et al.*[6]). A is called *ij*-θ-closed if A = *ij*-θ-*cl* A (Mukherjee *et al.*[6]). The complement of an *ij*-θ-closed set is called *ij*-θ-open, or equivalently, a set $B\,(\subset X)$ is *ij*-θ-open if for each $x \in B$, there is a Q_i-open set U such that $Q_j\text{-}cl\ U \subset B$. A set $A(\subset X)$ is called pairwise θ-closed (θ-open) if A is 12-θ-closed (12-θ-open) as well as 21-θ-closed (21-θ-open). To make this paper self-contained, we introduce as follows, a few more definitions and results.

Result 1.1[2, 6] — Let A, B be subsets of a space (X, Q_1, Q_2).

Then

(a) $A \subset B \Rightarrow ij\text{-}\theta\text{-}cl\ A \subset ij\text{-}\theta\text{-}cl\ B$.

(b) $Q_i\text{-}cl\ A \subset ij\text{-}\theta\text{-}cl\ A$.

(c) $A \in Q_j \Rightarrow Q_i\text{-}cl\ A = ij\text{-}\theta\text{-}cl\ A$.

(d) $ij\text{-}\theta\text{-}cl\,A = \bigcap \{Q_i\text{-}cl\,V : A \subset V \in Q_j\}$.

Definition 1.2[6] — A point x in a space (X, Q_1, Q_2) is called an *ij*-θ-adherent point of a net (x_α) with the directed set $(D, \geq)$ as the domain, if for each Q_i-open nbd U of x and each $\alpha \in D$, there exists $\beta \in D$ with $\beta \geq \alpha$ such that $x_\beta \in Q_j\text{-}cl\ U$. The net is said to *ij*-θ-converge to x if for every Q_i-open nbd U of x, there is $\alpha_0 \in D$ such that $x_\alpha \in Q_j\text{-}cl\ U$, for all $\alpha \in D$ with $\alpha \geq \alpha_0$.

Definition 1.3 — A space (X, Q_1, Q_2) is said to be

(a) *ij*-almost regular if for each Q_i-open set V in X and each $x \in V$, there exists a Q_i-open nbd U of x such that $Q_j\text{-}cl\, U \subset Q_i\text{-}int\ Q_j\text{-}cl\ V$,

(b) *ij*-regular if for each $x \in X$ and each Q_i-closed set A in X with $x \notin A$, there are a Q_i-open nbd U of x and a Q_j-open set V containing A such that $U \cap V = \phi$,

(c) pairwise almost regular (regular) if X is 12-almost regular (resp. 12-regular) and 21-almost regular (resp. 21-regular).

Remark 1.4 : Obviously an *ij*-regular space is *ij*-almost regular, but the converse is false[8].

The assertions in the following result are either known or easy to derive.

Result 1.5 — Let (X, Q_1, Q_2) be a space.

(a) The collection of all *ij*-θ-open sets in (X, Q_1, Q_2) forms a topology Q_i^θ on X such that $Q_i^\theta \subset Q_i$, for $i = 1$ and 2. If (X, Q_1, Q_2) is *ij*-regular then $Q_i = Q_i^\theta$.

(b) For any subset A of X, *ij*-θ-*cl* A is Q_i-closed, but not necessarily *ij*-θ-closed. If (X, Q_1, Q_2) is *ij*-almost regular, the *ij*-θ-*clA* is *ij*-θ-closed, so that a set is Q_i-closed iff it is *ij*-θ-closed.

2. Bitopological *QHC*-Spaces in Terms of Cover, Nets and Filterbases

We start by recalling the following definition of bitopological *QHC* spaces, first introduced in (Mukherjee[3]).

Definition 2.1 — A space (X, Q_1, Q_2) is called *ij-QHC* if every Q_j-open filterbase on X has an adherent point in (X, Q_i). The space X is said to be pairwise *QHC* if it is 12-*QHC* as well as 21-*QHC*.

To facilitate our discussion in the sequel, let us recall the following list of characterizations of an *ij-QHC* space, established in Mukherjee[3] and Mukherjee *et al.*[6].

Theorem 2.2 — For a space (X, Q_1, Q_2), the following are equivalent :

(a) X is *ij-QHC*.

(b) For every Q_i-open cover $\mathcal{U}$ of X, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $X = \bigcup \{Q_j\text{-}cl\, U : U \in \mathcal{U}_0\}$.

(c) Every net in X has an *ij*-θ-adherent point.

(d) If $\mathcal{U}$ is a collection of subsets of X, having the finite intersection property, then $\bigcap \{ij\text{-}\theta\text{-}cl\ U : U \in \mathcal{U}\} \neq \phi$.

We at once get the following corollary from (d) above :

Corollary 2.3 — In an *ij-QHC* space, every family of *ij*-θ-closed sets with finite intersection property has non-void intersection.

Definition 2.4[8] — A subset A of (X, Q_1, Q_2) is said to be an *ij*-regularly open set (*ij-ro*-set, for short) if $A = Q_i\text{-}int\ Q_j\text{-}cl\, A$, and the complement of an *ij-ro*-set is called an *ij*-regularly closed set or simply an *ij-rc*-set.

Theorem 2.5 — An *ij*-almost regular space (X, Q_1, Q_2) is *ij-QHC* iff every cover of X by *ij*-θ-open sets of X has a finite subcover.

PROOF : Let X be *ij-QHC* and $\mathcal{U}$ be a conver of X by *ij*-θ-open sets. Then for each $x \in X$, there is $U_x \in \mathcal{U}$ such that $x \in U_x$, and then $x \in V_x \subset Q_j\text{-}cl\, V_x \subset U_x$, for a Q_i-open set V_x. Now, $\{V_x : x \in X\}$ is a Q_i-open cover of X and hence by Theorem 2.2, $X = \bigcup_{i=1}^{n} Q_j\text{-}cl\, V_{x_i}$, for a finite subset $\{x_1, x_2, ..., x_n\}$ of X. Then $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. We incidentally note that for this part we do not need the *ij*-almost regularity of X.

Conversely, let X be *ij*-almost regular and $\mathcal{U}$ be a Q_i-open cover of X. Then $\mathcal{V} = \{Q_i\text{-}int\ Q_j\text{-}clU : U \in \mathcal{U}\}$ is a cover of X by *ij-ro*-sets. Since X is *ij*-almost regular, each member of $\mathcal{V}$ is clearly *ij*-θ-open. By hypothesis, there is a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $X = \bigcup \{Q_i\text{-}int\ Q_j\text{-}cl\ U : U \in \mathcal{U}_0\} \subset \bigcup \{Q_j\text{-}cl\, U : U \in \mathcal{U}_0\}$ proving that X is *ij-QHC*.

Definition 2.6 — A point x of a space (X, Q_1, Q_2) is called an *ij*-θ-complete adherent point of a subset A of X if for each *ij*-θ-open set U containing x, $|U \cap A| = |A|$, where for any subset B of X, $|B|$ denotes the cardinality of B.

Theorem 2.7 — An *ij*-almost regular space (X, Q_1, Q_2) is *ij-QHC* iff every infinite subset A of X has an *ij*-θ-complete adherent point in X.

PROOF : First suppose that the space X is *ij-QHC* and if possible, let A be an infinite subset of X without any *ij*-θ-complete adherent point in X. Then for each $x \in X$, there is an *ij*-θ-open set U_x containing x sucht hat

$$|U_x \cap A| < |A|. \qquad \text{... (1)}$$

Since $\{U_x : x \in X\}$ is a cover of X by *ij*-θ-open sets, by Theorem 2.5 we have, $X = \bigcup_{k=1}^{n} U_{x_k}$ for some finite subset $\{x_1, x_2, ..., x_n\}$ of X. Now, $A = \left(\bigcup_{k=1}^{n} U_{x_k}\right) \cap A = \bigcup_{k=1}^{n} (U_{x_k} \cap A)$ and hence

$$|A| = \max_{1 \le k \le n} \left\{ |U_{x_k} \cap A| \right\} \text{ which contradicts (1).}$$

Conversely, let an *ij*-almost regular space X be not an *ij-QHC* space. By Theorem

2.5, there is then a cover $\mathcal{U}$ of X by ij-θ-open sets such that for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, $\cup \mathcal{U}_0 \subsetneq X$. Let $r = \min \{ \mid \mathcal{V} \mid : \mathcal{V} \subset \mathcal{U}$ and $\mathcal{V}$ is a cover of $X\}$. It is clear that $r \geq N_0$ (= the cardinal number of the set of integers). Now, there exists $\mathcal{U}^* \subset \mathcal{U}$ such that $\mid \mathcal{U}^* \mid = r$ and $\mathcal{U}^*$ is a cover of X. We can well-order $\mathcal{U}^*$ by some minimal well-ordering $\leq$. Let us choose $U \in \mathcal{U}^*$ arbitrarily. The well-order $\leq$ being minimal, we must have $\mid \{V \in \mathcal{U}^* : V \leq U\} \mid < \mid \mathcal{U}^* \mid$. Furthermore, $\{V \in \mathcal{U}^* : V \leq U\}$ is not a cover of X, for each $U \in \mathcal{U}^*$, by definition of r. Corresponding to each $U \in \mathcal{U}^*$, we choose inductively a point $x_U \in X \setminus U$ $\{\{X_V\} \bigcup V : V \in \mathcal{U}^*$ and $V \leq U\}$ and consider the set $A = \{x_U : U \in \mathcal{U}^*\}$. It is easy to see that $x_U \neq x_V$ whenever $U, V \in \mathcal{U}^*$ with $U \neq V$. Thus $\mid A \mid = r > N_0$. It now suffices to show that A has no ij-θ-complete adherent point in X. For this let us consider an arbitrary point y of X. Since $\mathcal{U}^*$ is a cover of X by ij-θ-open sets, $y \in \mathcal{U}^*$ for some $U^* \in \mathcal{U}^*$, so U^* is an ij-θ-open set containing y. We now claim that $\{U : U \in \mathcal{U}^*$ and $x_U \in U^*\} \subset \{U : U \in \mathcal{U}^*$ and $U \leq U^*\}$. Indeed, for $U \in \mathcal{U}^*$ with $x_U \in U^*$ we obtain, by the choice of x_U, $U \leq U^*$. But the minimality of $\leq$ implies that $\mid \{U \in \mathcal{U}^* : U \leq U^*\} \mid < r$ so that $\mid A \bigcap U^* \mid < r = \mid A \mid$. This shows that y is not an ij-θ-complete adherent point of the infinite set A. This completes the proof of the theorem.

Theorem 2.8 — An ij-almost regular space (X, Q_1, Q_2) is ij-*QHC* iff every net in X with a well-ordered index set as domain, has an ij-θ-adherent point in X.

PROOF : The necessity part of the theorem follows immediately from Theorem 2.2 ((a) $\Rightarrow$ (c)). To prove the converse, we assume that A is an infinite subset of X. Let A be well-ordered by a minimal well-ordering. Then A may be assumed to be a net with a well-ordered index set as domain. By hypothesis, the net has an ij-θ-adherent point x (say) in X. It can easily be checked that for any Q_i-open nbd U of x, $|A \bigcap Q_j\text{-}cl\, U| = |A|$. Now, for any ij-θ-open set V containing x, there is a Q_i-open nbd U of x such that $Q_j\text{-}cl\ U \subset V$. Then $|A \bigcap V| = |A \bigcap Q_j\text{-}cl\, U| = |A|$. Hence x is an ij-θ-complete adherent point of A. The rest follows from the last theorem.

Remark 2.9 : We observe that the necessity parts of Theorems 2.5, 2.7 and 2.8 hold without the assumption of ij-almost regularity on the spaces under consideration.

We conclude this section by presenting an application of the results so far in the form of a 'fixed set theorem' for multifunctions on a ij-*QHC* space into itself.

Theorem 2.10 — Let F be a multifunction on an ij-*QHC* space X into itself such that $F(A)$ is ij-θ-closed for each ij-θ-closed set A in X. Then there is a non-void ij-θ-closed set A in X such that $F(A) = A$.

PROOF : Let $\mathcal{F} = \{A \subset X : A \neq \phi, A$ is ij-θ-closed and $F(A) \subset A\}$. Then $\mathcal{F} \neq \phi$ as $X \in \mathcal{F}$, and $\mathcal{F}$ is a poset with respect to set inclusion relation. Let $\mathcal{U}$ be a chain in $\mathcal{F}$ and let $\bigcap \mathcal{U} = U_0$. Since $\mathcal{U}$ is a family of ij-θ-closed sets with finite intersection property in the *ij-QHC* space X, U_0 is non-void by Corollary 2.3. Moreover, it is easy to see that U_0 is *ij*-θ-closed. Again, $F(U_0) \subset F(U) \subset U$, for each $U \in \mathcal{U}$. This implies that $F(U_0) \subset U_0$. Thus $U_0 \in \mathcal{F}$ and U_0 is a clearly a lower bound of $\mathcal{U}$. Then by Zorn's lemma, $\mathcal{F}$ has a minimal element A (say). Thus A is a non-void *ij*-θ-closed set and $F(A) \subset A$. But since $F(A)$ is *ij*-θ-closed and $F(F(A)) \subset F(A)$, we have $F(A) \in \mathcal{F}$. It then follows by the minimality of A that $F(A) = A$, and the theorem becomes proved.

Remark 2.11 : The existence of the fixed set A in the above theorem has been proved by use of Zorn's lemma and therefore, the theorem fails to give the actual method of construction of the fixed set A. But it is, at the same time, possible to construct such a non-void *ij*-θ-closed set under an additional condition as is shown below.

Theorem 2.12 — If, in addition to the conditions of Theorem 2.10, the graph $G(F) = \{(x, y) : x \in X, y \in Y \text{ and } y \in F(x)\}$ of F is *ij*-θ-closed, then the construction of a non-void *ij*-θ-closed set A such that $F(A) = A$, is possible.

PROOF : Since X is *ij*-θ-closed, $\{X, F(X), F^2(X), ...\}$ forms a sequence of non-void *ij*-θ-closed sets in X. Also, since $F(X) \subset X$, the sequence is decreasing and hence has the finite intersection property. By Corollary 2.3, $X \bigcap \left[\bigcap_{n=1}^{\infty} F^n(X) \right] = (= A, \text{ say}) \neq \phi$ also, A is clearly *ij*-θ-closed. For each $n = 2, 3, ...,$ we have $A \subset F^{n-1}(X)$ so that $F(A) \subset F^n(X)$, for $n = 1, 2, ...$ Then $F(A) \subset A$. In order to prove that $A \subset F(A)$, let $a \in A$. Then for each $n = 1, 2, ...,$ there exists a point $x_n \in F^n(X)$ such that $a \in F(x_n)$. In case $\{x_n : n = 1, 2, ...\}$ is a finite set, we have a positive integer m such that $x_n = x$ for all $n \geq m$; then $x \in A$ and $a \in F(x)$, i.e., $a \in F(A)$. Next suppose $\{x_n : n = 1, 2, ...\}$ is an infinite set in the *ij-QHC* space X. Then it has an *ij*-θ-adherent point $x_0 \in X$. Since for each n, $F^n(X)$ contains all but finitely many terms of the sequence, x_0 is an *ij*-θ-adherent point of $F^n(X)$, and hence $x_0 \in F^n(X)$ for each $n = 1, 2, ...$. Thus $x_0 \in A$ and (x_0, a) is an *ij*-θ-adherent point of $\{(x_n, a) : n = 1, 2, ...\} \subset G(F)$. Since $G(F)$ is *ij*-θ-closed, $a \in F(x_0)$ and hence $a \in F(A)$. Thus in either case $A \subset F(A)$.

3. BITOPOLOGICAL *QHC*-SPACES IN TERMS OF MULTIFUNCTIONS

Definition 3.1 — A multifunction F from a space X to a space Y will be called $Q_i P_i$ (Q_j)-lower (upper) θ*-continuous if for each Q_i-open set U in Y, $X \setminus F^-(U)$ (resp. $F^-(Y \setminus U)$) is *ij*-θ-closed in X, i.e., for each P_i-open set U in Y, $F^-(U)$ (resp. $F^+(U)$) is *ij*-θ-open in X, where as usual, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \bigcap B \neq \phi\}$, for any $B \subset Y$.

Now we give some equivalent formulations of the above types of multifunctions.

Lemma 3.2 — For a multifunction $F : (X, Q_1, Q_2) \to (Y, P_1, P_2)$, the following are equivalent :

(a) F is $Q_i P_i (Q_j)$-upper θ^*-continuous.

(b) For each $x \in X$ and each P_i-open set V in Y containing $F(x)$, there is an ij-θ-open set U in X containing x such that $F(U) \subset V$.

(c) For each $x \in X$ and each P_i-open set V in Y containing $F(x)$, there is a Q_i-open nbd U of x in X such that $F(Q_j\text{-}cl\, U) \subset V$.

PROOF : (a) $\Rightarrow$ (b) : Let $x \in X$ and V be a P_i-open set in Y containing $F(x)$. By (a), $F^+(V)$ is ij-θ-open, and clearly $x \in F^+(V)$. Choose $U = F^+(V)$. Then U is an ij-θ-open set containing x such that $F(U) = FF^+(V) \subset V$.

(b) $\Rightarrow$ (c) : For each $x \in X$ and each P_i-open set V in Y containing $F(x)$, there exists, by (b), an ij-θ-open set U in X containing x such that $F(U) \subset V$. Then there exists a Q_i-open nbd W of x such that $Q_j\text{-}cl\ W \subset U$. Hence $F(Q_j\text{-}cl\ W) \subset V$.

(c) $\Rightarrow$ (a) : If possible, let there exist a P_i-open set V in Y such that $F^+(V)$ is not ij-θ-open in X. Then there exists $y \in F^+(V)$ such that for each Q_i-open nbd U_y of y, Q_j-cl $U_y \not\subset F^+(V)$. Since $F(y) \subset V$ and V is P_i-open, there exists by (c), a Q_i-open nbd U of y such that $F(Q_j\text{-}cl\ U) \subset V$, i.e., Q_j-cl $U \subset F^+(V)$, which is a contradiction.

Lemma 3.3 — For a multifunction $F : X \to Y$, the following are equivalent :

(a) F is $Q_i P_i (Q_j)$-lower θ^*-continuous

(b) For each $x \in X$ and each P_i-open set V in Y with $x \in F^-(V)$ there is an ij-θ-open set U containing x such that $U \subset F^-(V)$.

(c) For each $x \in X$, and each P_i-open set V in Y with $x \in F^-(V)$, there is a Q_i-open set U in X such that $x \in U \subset Q_j\text{-}cl\ U \subset F^-(V)$.

PROOF : The proof is quite similar to that of Lemma 3.2 and is thus omitted.

We are now ready with the appliances to implicate them to the study of ij-QHC spaces.

Theorem 3.4 — For an ij-QHC space (X, Q_1, Q_2), the following are true :

(a) For each $Q_i P_i (Q_j)$-upper θ^*-continuous multifunction F on X to any space (Y, P_1, P_2), the multifunction F^* on X to Y given by $F^*(x) = ji\text{-}\theta\text{-}cl\ F(x)$, assumes a maximal value under set-inclusion relation.

(b) Each $Q_i P_i (Q_j)$ upper θ^*-continuous multifunction F on X to any space (Y, P_1, P_2) with ji-θ-closed point images asuumes a maximal value under set inclusion.

Moreover, if the space (X, Q_1, Q_2) is ij-almost regular, then each of the conditions (a) and (b) is a necessary and sufficient condition for the space X to be ij-QHC.

PROOF : (a) Let (X, Q_1, Q_2) be an ij-QHC space, and let the set $\mathcal{F} = \{F^*(x) : x \in X\}$ be partially ordered by inclusion. It is to be shown that $\mathcal{F}$ has a maximal element. For this we consider a linearly ordered subset L of $\mathcal{F}$. We choose an $y \in X$ such that $F^*(y) \in L$ and construct the set $B(y) = \{x \in X : F^*(y) \subset F^*(x)\}$. As L is a chain, $\mathcal{B} = \{B(y) : y \in X$ and $F^*(y) \in L\}$ is a filterbase on the ij-QHC space X. We claim that the members of $\mathcal{B}$ are ij-θ-closed. In fact, let $z \in ij\text{-}\theta\text{-}cl\ B(y)$ where

$B(y) \in \mathcal{B}$ and let W be a P_i-open set containing $F(z)$. Then by Lemma 3.2, there is a Q_i-open nbd V of z such that $F(Q_j\text{-}cl\ V) \subset W$. Since $z \in ij\text{-}\theta\text{-}cl\ B(y)$, we can find an $x^* \in Q_j\text{-}cl\ V \bigcap B(y)$. As $x^* \in B(y)$, we have $F^*(y) \subset F^*(x^*) = ji\text{-}\theta\text{-}cl\ F(x^*) \subset ji\text{-}\theta\text{-}cl\ F(Q_j\text{-}cl\ V) \subset ji\text{-}\theta\text{-}cl\ W = P_j\text{-}cl\ W$ (by Result 1.1 (c)). Thus $F^*(y) \subset P_j\text{-}cl\ W$, for every P_i-open set W containing $F(z)$. Hence $F^*(y) \subset \bigcap \{P_j\text{-}cl\ W : F(z) \subset W \in P_i\} = ji\text{-}\theta\text{-}cl\ F(z)$ (by Result 1.1 (d)) $= F^*(z)$ so that $z \in B(y)$. Hence $B(y)$ is ij-θ-closed and our claim has been established. Since $\mathcal{B}$ is a filterbase on the ij-QHC-space X, it now turns out by Theorem 2.2 that there is some $x_0 \in \bigcap \{ij\text{-}\theta\text{-}cl\ B(y) : B(y) \in \mathcal{B}\} = \bigcap \{B(y) : B(y) \in \mathcal{B}\}$. Then $F^*(x_0)$ is an upper bound of L. Hence by Zorn's lemma, $\mathcal{F}$ assumes a maximal element.

(a) $\Rightarrow$ (b) : It is obvious.

Let us now suppose that the space (X, Q_1, Q_2) is ij-almost regular. We prove that the space is ij-QHC if (b) holds. If possible, let X be not ij-QHC. Then by Theorem 2.8, there exists a net $S = \{S_\alpha\}$ with a well ordered set D as its domain, having no ij-θ-adherent point in X. Let D be given the order topology T and we consider the bitopological space (D, P_1, P_2), where $P_1 = P_2 = T$. Now, for each $\lambda \in D$, let us construct the set $A_\lambda = X \setminus ij\text{-}\theta\text{-}cl\ \{S_\alpha : \alpha \geq \lambda\}$. Now, given an $x \in X$, x being not an ij-θ-adherent point of S, there exists some $\lambda_0 \in D$ such that $x \notin ij\text{-}\theta\text{-}cl\ \{S_\alpha : \alpha \geq \lambda_0\}$ and hence $x \in A_{\lambda_0}$. Also, for any set B, ij-θ-$cl\ B$ is Q_i-closed and hence each A_λ is Q_i-open. Thus $\{A_\lambda : \lambda \in D\}$ is an increasing Q_i-open cover of X. Now for each $x \in X$, let us denote by $\lambda(x)$ the first suffix in D such that $x \in A_{\lambda(x)}$. Define $F : X \to D$ by $F(x) = \{\alpha \in D : \alpha \leq \lambda(x)\}$. Since (D, P_1, P_2) is pairwise regular, $F(x)$ is ji-θ-closed as well as ij-θ-closed for each $x \in X$ (vide Result 1.5(a)). We now show that F is $Q_i P_i (Q_j)$-upper θ^*-continuous. Let W be a P_i-open set containing $F(x)$ for an arbitrary $x \in X$. Since X is ij-almost regular, we have by Result 1.5(b) that $A_{\lambda(x)}$ is an ij-θ-open set containing x in X. Let $y \in A_{\lambda(x)}$. Then $\lambda(y) \leq \lambda(x)$ so that $F(y) \subset F(x) \subset W$. Thus $F(A_{\lambda(x)}) \subset W$. Hence F is $Q_i P_i (Q_j)$-upper θ^*-continuous by Lemma 3.2. But F cannot assume any maximal value under set-inclusion. For, if possible, let for some $x \in X$, $F(x)$ be maximal, then for every $y \in X$, we have $F(y) \subset F(x)$, i.e., $\lambda(y) \leq \lambda(x)$ and hence $A_{\lambda(y)} \subset A_{\lambda(x)}$. Thus $y \in A_{\lambda(x)}$ and as a result $X = A_{\lambda(x)}$ which is not possible. This completes the proof.

Definition 3.5 — A partial order relation '$\leq$' defined on a bitopological space (X, Q_1, Q_2) is said to be lower (upper) ij-θ^*-compatible if the set $\{x \in X : x \leq x_0\}$ (resp. $\{x \in X : x_0 \leq x\}$ is ij-θ-closed, for each $x_0 \in X$.

Definition 3.6 — A function f from a space (X, Q_1, Q_2) to a partially ordered set $(Y, \leq)$ is called lower (upper) ij-θ^*-continuous if $f^{-1}(\{y \in Y : y \leq y_0\})$ (resp. $f^{-1}(\{y \in Y : y_0 \leq y\}))$ is ij-θ-closed for each $y_0 \in Y$.

Theorem 3.7 — For an ij-QHC space X, the following are true :

(a) X has a maximal element with respect to each upper ij-θ^*-compatible partial order relation on X.

(b) Each upper ij-θ^*-continuous function f from X into a poset Z assumes a maximal value.

(c) Each $Q_i P_i$ (Q_j)-upper θ^*-continuous multifunction F from X into a space (Y, P_1, P_2), where (Y, P_i) is T_1, assumes a maximal value with respect to set-inclusion relation.

Furthermore, if the space (X, Q_1, Q_2) is ij-almost regular, then each of the conditions (a), (b) and (c) implies that X is ij-*QHC*, i.e., each of these conditions is equivalent to the statement that X is ij-*QHC*.

PROOF : (a) Let (X, Q_1, Q_2) be an ij-*QHC*-space and let '$\leq$' denote an upper ij-θ^*-compatible partial order relation on X. Suppose L denotes a linearly ordered subset of X and put $L_x = \{y \in X : x \leq y\}$, for each $x \in X$. By hypothesis, L_x is ij-θ-closed in X. Also, L being a chain, the family $\{L_x : x \in L\}$ has the finite intersection property. As X is ij-*QHC*, by Corollary 2.3 we can find a point $p \in \bigcap \{L_x : x \in L\}$ so that $p \geq x$, for all $x \in L$. Thus p is an upper bound of L. By Zorn's Lemma, X has then a maximal element with respect to the partial order relation in X.

(a) $\Rightarrow$ (b) : Let $(Z, \leq)$ be any poset and $f : X \to Z$ be any upper ij-θ^*-continuous function. We define a relation on X as follows :

$$\text{for } x, y \in X;\ x \leq y \text{ holds iff } f(x) \leq f(y) \text{ in } Z.$$

Clearly $(X, \leq)$ is a poset. Now, for each $x \in X, f^{-1}(\{z \in Z : f(x) \leq z\})$ $(=W_x$, say) is ij-θ-closed in X by upper ij-θ^*-continuity of f. But $W_x = \{y \in X : x \leq y\}$ so that the relation '$\leq$' on X is upper ij-θ^*-compatible, By (a), $(X, \leq)$ has a maximal element x_0 (say). By definition of $\leq$ on X, it follows that f assumes a maximal value at x_0 of X.

(b) $\Rightarrow$ (c) : The upper ij-θ^*-continuous multifunction F from X into the space (y, P_1, P_2) can be regarded as a single-valued function from X into the poset $(\mathcal{P}(Y) \setminus \{\phi\}, \subset)$, $\mathcal{P}(Y)$ being the power set of Y. It is enough to show that F treated in this sense is ij-θ^*-continuous. For this we need to s'ow that for each $B_0 \in \mathcal{P}(Y) \setminus \{\phi\}$ $(= \mathcal{P}^*(Y)$, say), $F^{-1}(\{B \in \mathcal{P}^*(Y) : B_0 \subset B\})$ is ij-θ-closed in X. Let $x \notin F^{-1}(\{B \in \mathcal{P}^*(Y) : B_0 \subset B\})$. Then $F(x) \neq B$, for any $B \in \mathcal{P}^*(Y)$ satisfying $B_0 \subset B$. Thus there is a $p \in B_0 \setminus F(x)$ and consequently, $F(x) \subset Y \setminus \{p\}$. (Y, P_i) being T_1, $\{p\}$ is P_i-closed in Y. Since the multifunction $F : X \to Y$ is upper ij-θ^*-continuous. $F^-(\{p\})$ is ij-θ-closed in X and clearly $x \notin F^-(\{p\})$. Thus there is a Q_i-open nbd U of x in X such that $Q_j\text{-}cl\, U \bigcap F^-(\{p\}) = \phi$, i.e., $p \in Y \setminus F(Q_j\text{-}cl\, U)$. Hence $Q_j\text{-}cl\, U \bigcap F^-(\{B \in \mathcal{P}^*(Y) : B_0 \subset B\}) = \phi$. Consequently, x is not an ij-θ-adherent point of $F^-(\{B \in \mathcal{P}^*(Y) : B_0 \subset B\})$ and so the latter set becomes ij-θ-closed.

Finally we assume that X is ij-almost regular and the condition (c) holds. Then proceeding similarly as in the proof of the last part of Theorem 3.4, it follows that

X is *ij-QHC*.

Theorem 3.8 — For an *ij-QHC*-space (X, Q_1, Q_2) the following statements hold :

(a) X has a minimal element with respect to each lower ij-θ^*-compatible partial order relation on X.

(b) Each lower ij-θ^*-continuous function $f : X \to Z$, where Z is a poset, assumes a minimal value.

(c) Each $Q_i P_i(Q_j)$-lower θ^*-continuous multifunction F from X into the P_i-closed subsets of any space (Y, P_1, P_2) assumes a minimal value with respect to set inclusion relation.

Moreover, under the assumption of ij-almost regularity of the space X, each of the conditions (a), (b) and (c) is a necessary and sufficient condition for X to be *ij-QHC*.

PROOF : Since the proof of (a) and (a) $\Rightarrow$ (b) are quite similar to the corresponding ones of Theorem 3.7, we take up the proof of (b) $\Rightarrow$ (c) as follows.

As in the proof of (b) $\Rightarrow$ (c) of Theorem 3.7, we treat F to be a function from X to the poset $(Y^*, \subset)$, where Y^* denotes the set of all nonvoid P_i-closed sets in Y, and show that the function $F : X \to Y^*$ is lower ij-θ^*-continuous. For that we need to show that for each $B_0 \in Y^*$, $F^{-1}(\{B \in Y^* : B \subset B_0\})$ is ij-θ-closed in X. Let $x \in X \setminus F^{-1}(\{B \in Y^* : B \subset B_0\})$. Then for each $B \subset B_0$,

$$F(x) \neq B, \text{ i.e., } F(x) \bigcap (Y \setminus B_0) \neq \phi \qquad \dots (1)$$

Now, since the multifunction $F : (X, Q_1, Q_2) \to (Y, P_1, P_2)$ is lower ij-θ^*-continuous and B_0 is P_i-closed in Y, it follows that $X \setminus F^-(Y \setminus B_0)$ is ij-θ-closed in X. But (1) implies that $x \notin X \setminus F^-(Y \setminus B_0)$. Hence there exists a Q_i-open nbd U of x such that $Q_j\text{-}cl\ U \bigcap (X \setminus F^-(Y \setminus B_0)) = \phi$, i.e., $Q_j\text{-}cl\ U \subset F^-(Y \setminus B_0)$. Hence for each $y \in Q_j\text{-}cl\ U$, $y \in F^{-1}(Y \setminus B_0)$, i.e., $F(y) \bigcap (Y \setminus B_0) \neq \phi$, for all $B \in Y^*$ satisfying $B \subset B_0$. This shows that $Q_j\text{-}cl\ U \bigcap F^{-1}(\{B \in Y^* : B \subset B_0\}) = \phi$, i.e., $x \notin ij\text{-}\theta\ cl\ F^{-1}(\{B \in Y^* : B \subset B_0\})$ so that $F^{-1}(\{B \in Y^* : B \subset B_0\})$ is ij-θ-closed in X, and we have established the required result. By (b), F then assumes a minimal value with respect to set inclusion.

Lastly, suppose that X is ij-almost regular and condition (c) holds. We prove that X is *ij-QHC*. If not, there exists by Theorem 2.8, a net $S = \{S_\alpha\}$ with a well-ordered set D as its domain and having no ij-θ-adherent point in X. Consider the space (D, P_1, P_2), where $P_1 = P_2$ = the order topology on D. For each $\beta \in D$, if we define $A_\beta = X \setminus ij\text{-}\theta\text{-}cl(\{S_\alpha : \alpha \in D \text{ and } \alpha \geq \beta\})$, then as in the proof of Theorem 3.4 it follows that $\{A_\beta : \beta \in D\}$ is an increasing Q_i-open cover of X. Let β_x be the first element in D such that $x \in A_{\beta_x}$. Let us now define a multifunction $F : X \to D$ by $F(x) = \{\alpha \in D : \alpha \geq \beta_x\}$. Then using Lemma 3.3, we can ultimately show that F is a $Q_i P_i(Q_j)$-lower θ^*-continuous multifunction from X into the P_i-closed subsets of the space (D, P_1, P_2) assuming no minimal value with respect to the set inclusion relation.

Remark 3.9 : Since a pairwise Urysohn *ij-QHC* space is necessarily *ij*-almost regular Mukherjee *et al.*[6], it follows that a pairwise Urysohn space is *ij-QHC* iff any one of the conditions (a) and (b) of Theorem 3.4 or any one of (a), (b) and (c) of Theorem 3.7 or Theorem 3.8 holds.

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