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CONCERNING BITOPOLOGICAL *QHC* EXTENSIONS

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In this paper our main task is to show the existence of one point pairwise *QHC* extensions of a non pairwise *QHC* space. Two such extensions are constructed which are shown to be respectively projectively maximum and minimum in the class of all such one point pairwise *QHC* extensions. For this purpose the notion of locally pairwise *QHC* spaces are introduced. It is shown, in addition, that the classical set-topological characterization of a locally *QHC* space in terms of the remainders of arbitrary *QHC* extensions fails here. Nevertheless, it has been possible to establish one side of the characterization for any bitopological space.

Apart from the famous Katětov's *H*-closed extension of a non-*H*-closed space, there are available methods for constructing one-point *H*-closed extensions of locally *H*-closed topological spaces. The papers of Girou[1], Obreanu[7], Porter and Votaw[9, 10], Tikoo[13] and Porter[8] deserve mentioning in this connection. Obreanu[7] proved the existence of a projective maximum and a projective minimum for one-point *H*-closed extensions. The problem was also tackled by Porter[8]. To our knowledge, such questions of getting one-point bitopological *H*-closed or quasi *H*-closed (*QHC*, for short) extensions have not been considered so far, although bitopological *QHC*-spaces, under the terminology *ij-QHC* and pairwise *QHC*-spaces, have been studied rather exhaustively[3-6]. It is thus our endeavour here to deal with the problem of studying one-point *QHC* extension of a locally *QHC*-space in bitopological setting. For this we required to frame a suitable definition of a bitopological locally *QHC*-space, termed pairwise locally *QHC*-space. We have thus been able to ultimately prove the existence of a projective maximum and a projective minimum for one-point pairwise *QHC* extensions of a pairwise Hausdorff, pairwise locally *QHC*-space. Finally, we have studied briefly the question of characterizing a pairwise locally *QHC*-space in terms of the remainders of pairwise *QHC* extensions. Such a question for a topological space has been considered an analysed by many mathematicians, one recently obtained formulation being that a topological space is locally *H*-closed iff any *H*-closed extension has a θ-closed remainder. We have seen that the bitopological counterpart of such a formulation in our context is false. However, we have obtained one part of the characterization and have supplied a counterexample for the other part.

Throughout the paper, by spaces X and Y we shall mean the bitopological spaces (X, Q_1, Q_2) and (Y, P_1, P_2) respectively, where Q_1, Q_2 (resp. P_1, P_2) are topologies on X (resp. on Y). Whenever i and j will both occur in any context, we assume that i, j = 1, 2 and $i \neq j$. For a subset A in a space (X, Q_1, Q_2), Q_i-int A and Q_i-cl A will respectively mean the interior and closure of A in (X, Q_i), for i = 1, 2. A point $x \in X$ is said to be in the ij-θ-closure of a subset A of X, denoted by $x \in ij$-θ-cl A (Kariofillis[3]) if for every Q_i-open neighbourhood (henceforth nbd, for short) U of x, Q_j-cl $U \cap A \neq \phi$. $A(\subset X)$ is called ij-θ-closed if A = ij-θ-cl A Mukherjee *et al.*[6]. A space (X, Q_1, Q_2) is called pairwise Hausdorff[2] if for any two distinct points x, y of X, there exist a Q_1-open nbd U of x and a Q_2-open nbd V of y such that $U \cap V = \phi$. $A(\subset X)$ is said to be an ij-regularly open set[12] (ij-ro-set, for short) if A = Q_i-int Q_j-cl A and complement of an ij-ro-set is called an ij-regularly closed set (ij-rc-set, for short). A point x in (X, Q_1, Q_2) is called an ij-θ-adherent point of a filterbase $\mathcal{F}$ on X (Kariofillis[3]) if $x \in \bigcap$ {ij-θ-cl F : $F \in \mathcal{F}$}. The set of all ij-θ-adherent points of $\mathcal{F}$ is called the ij-θ-adherence of $\mathcal{F}$ and is denoted by ij-θ-ad $\mathcal{F}$.

Definition 1 — A space (X, Q_1, Q_2) is called ij-QHC (Mukherjee[4]) if every Q_j-open filterbase on X has a Q_i-adherent point; equivalently, if every Q_i-open cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{U}_0$ such that $X = \bigcup$ {Q_j-cl U : $U \in \mathcal{U}_0$} (Mukherjee[4]).

X is called pairwise QHC if it is 12-QHC as well as 21-QHC.

Definition 2 (Mukherjee *et al.*[6]) — A set A in a space (X, Q_1, Q_2) is called an ij-H-set if for each cover of A by Q_i-open sets of X, there is a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subset \bigcup$ {Q_j-cl U : $U \in \mathcal{U}_0$}. A is called a pairwise H-set if it is a 12-H-set and a 21-H-set.

Remark 3 : It is clear that every ij-QHC subspace A (i.e., when $(A, (Q_1)_A, (Q_2)_A)$ is ij-QHC) of a space X is always an ij-H-set. That the converse is false was shown in (Sen and Banerjee[11]). Moreover, it is proved in Mukherjee *et al.*[6] that a subset A of a space (X, Q_1, Q_2) is an ij-H-set iff for each filterbase $\mathcal{F}$ on A, (ij-θ-ad $\mathcal{F}$) $\cap A \neq \phi$.

According to Mukherjee and Banerjee[5], a subset A of a space (X, Q_1, Q_2) is said to satisfy 'condition C_{ij}' if Q_i-cl (Q_j-int A) $\subset Q_j$-cl (Q_j-int A). They have also proved that :

Theorem 4 — In an ij-QHC space (X, Q_1, Q_2), an ij-rc-set A is ij-QHC if A satisfies condition C_{ij}.

Throughout the deliberations in this section, we shall need to assume, as is the case in its topological counterpart, that every ij-rc-set A in an ij-QHC space (X, Q_1, Q_2) is an ij-QHC sub-set, i.e., $(A, (Q_1)_A, (Q_2)_A)$ is ij-QHC. As we have already noticed in Theorem 4, this requirement is always met if the Condition C_{ij} is assumed.

Thus we shall henceforth assume such a condition to hold for all spaces under consideration.

We now quote the definition of extension of a bitopological space as proposed in Mukherjee[4].

Definition 5 — A bitopological space (X^*, Q_1^*, Q_2^*) will be called an extension of a bitopological space (X, Q_1, Q_2) if $X \subset X^*$, $Q_i^*\text{-}cl\ X = X^*$ and $(Q_i^*)_X = Q_i$, for $i = 1, 2$. If, in addition, (X^*, Q_1^*, Q_2^*) is *ij-QHC* (pairwise *QHC*) then it will be called an *ij-QHC* (respectively pairwise *QHC*) extension of (X, Q_1, Q_2).

Let us now set the following definitions :

Definition 6 — An extension (Y, P_1, P_2) of a space (X, Q_1, Q_2) will be called a one-point *ij-QHC* (pairwise *QHC*) extension of a space (X, Q_1, Q_2) if (Y, P_1, P_2) is *ij-QHC* (resp. pairwise *QHC*) and $(Y \setminus X)$ consists of a single point.

Definition 7 — A space (X, Q_1, Q_2) is said to be an *ij*-locally *QHC* space if every point of X has a Q_j-open nbd U such that $Q_i\text{-}cl\ U$ is an *ij-H*-set in X; the space X is called pairwise locally *QHC* if it is 12-as well as 21-locally *QHC*.

Example 8 — Let R denote the set of real numbers and let Q_1, Q_2 respectively denote the usual and discrete topologies on R. Then (R, Q_1, Q_2) is neither 12-*QHC* nor 21-*QHC*, but is 12-locally *QHC* without being 21-locally *QHC*.

Remark 9 : It is clear that an *ij-QHC* space is *ij*-locally *QHC*. But it is at the same time true that the *ij-QHC* property of a space (X, Q_1, Q_2) need not imply that the space is *ji*-locally *QHC*. In fact, if X denotes the set of reals with the topologies Q_1, Q_2 respectively given by Q_1 = usual topology on X and $Q_2 = \{U \subset X : O \notin X$ or $(X \setminus U)$ is finite$\}$, then (X, Q_1, Q_2) is 21-*QHC* but not 12-locally *QHC*.

Theorem 10 — Let a pairwise Hausdorff space (Y, P_1, P_2) be a one-point *ij-QHC* extension of a space (X, Q_1, Q_2). Then

(a) (X, Q_1, Q_2) is an *ij*-locally *QHC* space.

(b) $X \in P_1 \cap P_2$.

PROOF : (a) Let $Y \setminus X = \{r\}$. Since (Y, P_1, P_2) is pairwise Hausdorff, for each $x \in X$ there is a P_j-open nbd U_x of x such that $r \notin P_i\text{-}cl\ U_x$. Since (Y, P_1, P_2) is *ij-QHC* and $P_i\text{-}cl\ U_X$ is an *ij-rc* set, it follows that $P_i\text{-}cl\ U_x$ is *ij-QHC* and hence an *ij-H*-set in X. Thus X is *ij*-locally *QHC*.

(b) Follows clearly by the pairwise Hausdorffness of (Y, P_1, P_2).

In the next theorem we shall give a method for obtaining a pairwise *QHC* extension of a pairwise locally *QHC* space. The following lemma leads us half way to this contention.

Lemma 11 — Let (X, Q_1, Q_2) be an *ij*-locally *QHC* space which is not *ij-QHC* and let $X^* = X \cup \{p\}$ where $p \in X$. Then

(a) the Q_i-open filter $\mathcal{G}_i$ (say) generated by $\{U \subset X : (X \setminus U)$ is an *ij-H*-set in $X\}$ is Q_j-free,

(b) $Q_i^* = Q_i \bigcup \{\{p\} \bigcup U : U \in \mathcal{G}_i\}$ is a topology on X^*.

PROOF : (a) Let, if possible, x be a Q_j-adherent point of $\mathcal{G}_i$. Now since X is ij-locally QHC, there is a Q_j-open nbd U of x such that Q_i-$cl\ U$ is an ij-H-set in X. Then $X \setminus Q_i\text{-}cl\ U \in \mathcal{G}_i$ and $U \bigcap (X \setminus Q_i\text{-}cl\ U) = \phi$, which is a contradiction.

(b) Straightforward and omitted.

Theorem 12 — Let (X, Q_1, Q_2) be a pairwise locally QHC space which is neither 12-QHC nor 21-QHC. Then there exists a one-point pairwise QHC extension of (X, Q_1, Q_2).

PROOF : Let X^*, Q_1^* and Q_2^* be constructed as in Lemma 11. Then clearly (X^*, Q_1^*, Q_2^*) is a one-point extension of (X, Q_1, Q_2). Now let $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ be a Q_i^*-open cover of X^*. There is a B_0 in $\mathcal{B}$ containing p and hence there is a $U \in \mathcal{G}_i$ such that $\{p\} \bigcup U \subset B_0$ and $(X \setminus U)$ is an ij-H-set in X. There is a finite subfamily $\{B_0, B_1, ..., B_n\}$ of $\mathcal{B}$ such that $X \setminus U \subset \bigcup_{k=1}^{n} Q_j^*\text{-}cl\, B_k$. Now, $X^* \subset B_0 \bigcup \left(\bigcup_{k=1}^{n} Q_j^*\text{-}cl\, B_k \right)$, i.e., $X^* \subset \bigcup_{k=0}^{n} Q_j^*\text{-}cl\, B_k$. Hence (X^*, Q_1^*, Q_2^*) is ij-QHC. Now putting (1, 2) and (2, 1) successively for the pair (i, j), we see that (X^*, Q_1^*, Q_2^*) is pairwise QHC.

Remark 13 : It is easy to see that (X^*, Q_1^*, Q_2^*) becomes pairwise Hausdorff whenever (X, Q_1, Q_2) is so.

We next show that the above one-point pairwise QHC extension has a unique feature in the light of the following definition.

Definition 14 — Let (Y, P_1, P_2) and (Z, R_1, R_2) be two extensions of a space (X, Q_1, Q_2). The space (Y, P_1, P_2) is said to be projectively larger than (Z, R_1, R_2) or, equivalently, (Z, R_1, R_2) is said to be projectively smaller than (Y, P_1, P_2), if there is a bicontinuous function f from Y onto Z (i.e., a surjection $f : Y \to Z$ such that $f : (Y, P_i) \to (Z, R_i)$ is continuous for i = 1, 2) which leaves X pointwise fixed.

Theorem 15 — The one point pairwise QHC extension (X^*, Q_1^*, Q_2^*) of a pairwise Hausdorff space (X, Q_1, Q_2), as constructed in Lemma 11 and Theorem 12, is a projective minimum in the set of all one point pairwise QHC pairwise Hausdorff extensions of the space (X, Q_1, Q_2).

PROOF : Let (Y, P_1, P_2) be a one-point pairwise QHC extension of a pairwise Hausdorff space (X, Q_1, Q_2) and let $Y \setminus X = \{r\}$. Define $f : Y \to X^*$ by putting $f(x) = x$, for all $x \in X$ and $f(r) = p$.

Clearly f is bijective. By Theorem 10 and Remark 13, X is Q_i^*-open in X^* and P_i-open in Y. Hence it follows that $f : (Y, P_i) \to (X^*, Q_i^*)$ is continuous for all $x \in X$. Now let U be a Q_i^*-open nbd of p. Then there exists a $V \in \mathcal{G}_i$ such that $X \setminus V$ is an *ij*-*H*-set in X, $U \supset \{p\} \bigcup V$ and $X \setminus V$ is P_i-closed in Y. Thus f^{-1} $[\{p\} \bigcup V] = Y \setminus (X \setminus V)$ is P_i-open in Y containing r and is contained in $f^{-1}(U)$, so that $f : (Y, P_i) \to (X^*, Q_i^*)$ is continuous at r as well. Hence f is continuous from (Y, P_i) to (X^*, Q_i^*), for $i = 1, 2$. Thus (X^*, Q_1^*, Q_2^*) is a projective minimum.

Theorem 16 — Let (X, Q_1, Q_2) be an *ij*-locally *QHC* pairwise Hausdorff space, and $B \subset X$.. Then Q_i-*cl* B is *ji*-locally *QHC* if there exists some *ij*-*QHC* extension (Y, P_1, P_2) (say) of X such that P_i-*cl* B is *ji*-*QHC*.

PROOF : Suppose that there exists some *ij*-*QHC* extension of X, say (Y, P_1, P_2) such that P_i-*cl* B is a *ji*-*QHC* subset of Y. Consider the projectively minimum one-point pairwise Hausdorff extension (X^*, Q_1^*, Q_2^*) of X, as considered in the above theorem. Then there exists a bi-continuous function $f : (Y, P_1, P_2) \to (X^*, Q_1^*, Q_2^*)$ which leaves X pointwise fixed. We claim that $f(P_i\text{-}cl\, B) = Q_i^*\text{-}cl\, B$. In fact, firstly we have $f(P_i\text{-}cl\, B) \subset Q_i^*\text{-}cl\, f(B) = Q_i^*\text{-}cl\, B$. Again, since *bi*-continuous image of an *ij*-*QHC* space is *ij*-*QHC*, for $i, j = 1, 2$ $(i \neq j)$ (vide Mukherjee and Banerjee[5]), we have $f(P_i\text{-}cl\, B)$ is a *ji*-*QHC* subset of X^*. Moreover, it is known[6] that any *ij*-*H*-set in a pairwise Hausdorff space (X, Q_1, Q_2) is Q_j-closed. Thus $f(P_j\text{-}cl\, B)$ is Q_i^*-closed in X^* and contains $f(B)$, so that $Q_i^*\text{-}cl\, f(B) \subset f(P_i\text{-}cl\, B)$, and hence Q_i^*-*cl* B is a *ji*-*QHC* subset of X^*. But Q_i^*-*cl* B is then Q_i-*cl* B or a one-point *ji*-*QHC* extension of Q_i-*cl* B. So Q_i-*cl* B must be a *ji*-locally *QHC* space, by Theorem 10.

Our next task is to construct a one-point pairwise *QHC* extension which is a projective maximum in the set of all one-point pairwise *QHC* pairwise Hausdorff extensions of the space under consideration. We shall need the following lemmas to that end.

Lemma 17 — If every Q_i-open set in (X, Q_1, Q_2) satisfies Condition C_{ji}, then $(Q_i\text{-}int\ Q_j\text{-}cl\ U) \bigcap (Q_i\text{-int}\ Q_j\text{-}cl\ V) = Q_i\text{-int}\ Q_j\text{-}cl\ (U \bigcap V)$ for any $U, V \in Q_i$.

PROOF : Since $Q_i\text{-int}\ Q_j\text{-}cl\ (U \bigcap V) \subset Q_i\text{-int}\ Q_j\text{-}cl\ U \bigcap Q_i\text{-int}\ Q_j\text{-}cl\ V$, it suffices to show that $Q_i\text{-int}\ Q_j\text{-}cl\ U \bigcap Q_i\text{-int}\ Q_j\text{-}cl\ V \subset Q_i\text{-int}\ Q_j\text{-}cl\ (U \bigcap V)$, i.e., $Q_i\text{-int}\ (Q_j\text{-}cl\ U \bigcap Q_j\text{-}cl\ V) \subset Q_i\text{-int}\ Q_j\text{-}cl\ (U \bigcap V)$.

If not, there exists a Q_i-open set W such that $W \subset Q_j\text{-}cl\ U \bigcap Q_j\text{-}cl\ V$, but $W \not\subset Q_j\text{-}cl\, (U \bigcap V)$.

Let $$A = W \setminus Q_i\text{-}cl\ (U \bigcap V). \qquad \ldots (1)$$

Then A is Q_i-open. Obviously $U \bigcap V \in Q_i$ and by Condition C_{ji}, $Q_j\text{-}cl\, (U \bigcap V) \subset Q_i\text{-}cl\ (U \bigcap V)$. Hence $A \subset (Q_j\text{-}cl\, U \bigcap Q_j\text{-}cl\, V)\ Q_j\text{-}cl\, (U \bigcap V)$. Since $A \subset Q_j\text{-}cl\, U \subset Q_i\text{-}cl\, U$ (by Condition C_{ji}), we have $A \bigcap U \neq \phi$. Let $y \in U \bigcap A$. Since $y \in A$

$\subset Q_j\text{-}cl\, V \subset Q_i\text{-}cl\, V$ (by Condition C_{ji}) and $U \cap A$ is a Q_i-open nbd of y, we have $U \cap A \cap V \neq \phi$. ... (2)

But from (1) we have $A \cap Q_i\text{-}cl\,(U \cap V) = \phi$. Hence $A \cap U \cap V = \phi$ which contradicts (2). Hence the result is proved.

Lemma 18 — Let (X, Q_1, Q_2) be a space in which every Q_j-open set satisfies Condition C_{ij}. Let $Y(\subset X)$ be an *ij-H*-set in X, and $A(\subset Y)$ be an *ij-rc*-set in X. Then A is an *ij-QHC* subset.

PROOF : Let $\mathcal{F}$ be a $(Q_j)_A$-open filterbase on A. Let $A = Q_i\text{-}cl\, U$, where $U \in Q_j$. We consider the collection $\mathcal{G} = \{F \cap U : F \in \mathcal{F}\}$. It is straightforward to check that $\mathcal{G}$ is a Q_j-open filterbase on X. Then $(Q_i\text{-}ad\ \mathcal{G}) \cap Y \neq \phi$, as Y is an *ij-H*-set in X. Thus

$$\phi \neq [\bigcap \{Q_i\text{-}cl\,(F \cap U) : F \in \mathcal{F}\}] \cap Y$$

$$= [\bigcap \{(Q_i)_A\text{-}cl\ (F \cap U) : F \in \mathcal{F}\}] \cap Y \text{ (as } A \text{ is } Q_i\text{-closed)}$$

$$\subset \bigcap \{(Q_i)_A\text{-}cl\, F : F \in \mathcal{F}\} = [(Q_i)_A\text{-}ad\ \mathcal{F}].$$

Hence $(A, (Q_1)_A, (Q_2)_A)$ is *ij-QHC*.

Lemma 19 — Let (X, Q_1, Q_2) be an *ij*-locally *QHC*-space which is not *ij-QHC*. Further suppose that every Q_i-open set in X satisfies Condition C_{ji}, for $i, j = 1$ and 2 $(i \neq j)$. Then

(a) $\mathcal{F}_i = \{U \in Q_i : X \setminus Q_i\text{-int}\, Q_j\text{-cl } U$ is an *ij-H*-set in $X\}$ is a Q_j-free filter,

(b) if $X^* = X \cup \{p\}$, where $p \notin X$, then $Q_i^* = Q_i \cup \{\{p\} \bigcup U : U \in \mathcal{F}_i\}$ is a topology on X^*.

PROOF : (a) It is a routine work, in view of Lemmas 18 and 19, to check that $\mathcal{F}_i$ is indeed a filter on X. Let, if possible x be a Q_j-adherent point of $\mathcal{F}_i$. Now, since X is an *ij*-locally *QHC* space there is a Q_j-open nbd V of x such that $Q_i\text{-}cl\, V$ is an *ij-H*-set, i.e., $X \setminus [Q_i\text{-int}\ Q_j\text{-}cl\ (X \setminus Q_i\text{-}cl\ V)]$ is an *ij-H*-set in X, and hence $X \setminus Q_i\text{-}cl\ V \in \mathcal{F}_i$. Since $V \cap (X \setminus Q_i\text{-}cl\ V) = \phi$, we arrive at a contradiction.

(b) It is straightforward in view of (a) above.

Remark 20 : In the above Lemma, if (X, Q_1, Q_2) is pairwise Hausdorff, then so is (X^*, Q_1^*, Q_2^*).

Theorem 21 — Let (X^*, Q_1^*, Q_2^*) be the one-point extension of the space (X, Q_1, Q_2) as constructed in Lemma 19 under the conditions stated therein. Then

(a) (X^*, Q_1^*, Q_2^*) is pairwise *QHC*,

(b) If, in addition, (X, Q_1, Q_2) is pairwise Hausdorff, then (X^*, Q_1^*, Q_2^*) is a projective maximum in the set of one-point pairwise Hausdorff pairwise *QHC* extensions of (X, Q_1, Q_2).

PROOF : (a) Let $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ be a Q_i^*-open cover of X^*. There is a B_{α_0} in $\mathcal{B}$ containing p. Hence there is a $U \in \mathcal{F}_i$ such that $\{p\} \bigcup U = B_{\alpha_0}$ and $X \setminus Q_i$-int Q_j-cl U is an ij-H-set in X. So there is a finite subfamily $\{B_{\alpha_1}, ..., B_{\alpha_n}\}$ of $\mathcal{B}$ such that $X \setminus Q_i\text{-int } Q_j\text{-}cl\, U \subset \bigcup_{k=1}^{n} Q_j^* \text{-}cl\ B_{\alpha_k}$. Also, $\{p\} \bigcup (Q_i\text{-int } Q_j\text{-}cl\ U) \subset \{p\} \bigcup Q_j\text{-}cl\ U \subset Q_j^*\text{-}cl\ B_{\alpha_0}$. So $X^* \subset Q_j^*\text{-}cl\ B_{\alpha_0} \bigcup \left(\bigcup_{k=1}^{n} Q_j^*\text{-}cl\, B_{\alpha_k} \right)$, i.e., $X^* \subset \bigcup_{k=0}^{n} Q_j^*\text{-}cl\, B_{\alpha_k}$. This shows that (X^*, Q_1^*, Q_2^*) is 12-*QHC* as well as 21-*QHC*, and hence is pairwise *QHC*.

(b) Let (Y, P_1, P_2) be a one-point pairwise *QHC* extension of the pairwise Hausdorff space (X, Q_1, Q_2) and let $Y \setminus X = \{r\}$. Define $f : (X^*, Q_1^*, Q_2^*) \to (Y, P_1, P_2)$ by putting $f(x) = x$, for all $x \in X$, and $f(p) = r$. Clearly f is bijective. Since X is Q_i^*-open in X^* and P_i-open in Y (by Theorem 10), it follows that $f : (X^*, Q_i^*) \to (Y, P_i)$ is continuous for all $x \in X$ and for $i = 1, 2$. Now, let U be a P_i-open nbd of r. Since Y is ij-*QHC*, $Y \setminus P_i$-int P_j-cl U is also ij-*QHC* with respect to its subspace topologies. Again, $U \bigcap X (= W$, say$) \in Q_i$, we have P_i-int P_j-cl U = $(Q_i$-int Q_j-cl $W) \bigcup \{r\}$ and $Y \setminus P_i$-int P_j-cl U = $X \setminus Q_i$-int Q_j-cl W is an ij-H-set in X. Thus $f^{-1}(U) = W \bigcup \{p\}$ is a Q_i^*-open nbd of p. This establishes the continuity of f from (X, Q_i^*) to (Y, P_i) at p. Hence $f : (X, Q_i^*) \to (Y, P_i)$ is continuous for i = 1, 2, and consequently, (X^*, Q_1^*, Q_2^*) becomes a projective maximum.

In a topological space, different conditions necessary or sufficient or both, for a given subspace of the space to be locally H-closed, are known. One such condition for the whole space in terms of the remainders of H-closed extensions states (vide Girou[1]) that a topological space X is locally H-closed iff the remainder $X^* \setminus X$ is θ-closed in every H-closed extension X^* of X. We now show that only one part of the corresponding form of this result in our case of bitopological setting, is true. The following lemma practically proves our contention, and incidentally gives a sufficient condition for a subspace of a bitopological space to be locally ij-*QHC*.

Lemma 22 — A subset B of a space (X, Q_1, Q_2) is ij-locally *QHC* if X is ij-*QHC* and $X \setminus B$ is ji-θ-closed in X.

PROOF : Let $x \in B$. Then $x \notin X \setminus B$, so that there exists a Q_j-open nbd U of x such that Q_i-cl $U \bigcap (X \setminus B) = \phi$ which gives $Q_i\text{-}cl\, U \subset B$. Since Q_i-cl U is an ij-rc-set in the ij-*QHC* space X, Q_i-cl U is an ij-*QHC* subset containing x. Thus U is a

$(Q_j)_B$-open nbd of x such that $(Q_i)_B$-$cl\ U$ is an ij-H-set in B, and consequently, B is ij-locally QHC.

Theorem 23 — A space (X, Q_1, Q_2) is ij-locally QHC if the remainder $X^*\backslash X$ is ji-θ-closed in an ij-QHC extension (X^*, Q_1^*, Q_2^*) of X.

PROOF : Follows at once from Lemma 22.

As we have already remarked, the converse of the above theorem is not true, in general, for bitopological spaces, although the converse is true for topological space.

Example 24 — Let Y denote the set of real numbers endowed with the topologies P_1 and P_2, where P_1 and P_2 respectively stand for the usual topology and the cofinite topology on Y. Let X be the set of all rational numbers with the topologies Q_1 and Q_2, given by $Q_i = (P_i)_X$, for i = 1, 2. Then (Y, P_1, P_2) is a pairwise QHC extension of (X, Q_1, Q_2), where the latter space is clearly a pairwise locally QHC subspace. But the remainder $Y\backslash X$ is neither 12-θ-closed nor 21-θ-closed.

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