Indian J. pure appl. Math., **27**(3) : 265-275, March 1996

MODIFIED BESSEL TRANSFORM ON THE SPACES LG'_α

STEVAN PILIPOVIĆ AND MIRJANA STOJANOVIĆ

Institute of Mathematics, University of Novi Sad,
Trg D. Obradovića 4, 21000 *Novi Sad, Yugoslavia*

(*Received* 12 *August* 1994; *accepted* 25 *October* 1995)

We investigate the modified Bessel transform on the spaces LG'_α, $\alpha > -1$, by using the Laguerre expansion of their elements. We find the operational formula for this transform and apply it in solving partial differential equations of the form

$$P_1\left(\frac{\partial^2}{\partial x^2}x+\frac{\alpha}{x}\right)P_2\left(\frac{\partial}{\partial y}\right)u(x,y)=f(x,y)$$

where $[0,\infty)\in y\rightarrow f(\cdot,y)\in LG'_\alpha$ is k-times differentiable mapping and P_1 and P_2 are polynomials with constant coefficients.

1. INTRODUCTION

Integral transforms connected with the Bessel functions and Bessel equations on the spaces of generalized functions have been studied intensively, recently. There are several definitions of the Hankel transform of certain classes of generalized functions[1-3, 5, 7, 10].

Duran[4] defined the Hankel transform on the space LG'_α, $\alpha > -1$. This transform for LG'_α has the same role as the Fourier transform for tempered distribution. If $f=\sum_{n=0}^{\infty} a_{n\alpha}\, l_{n\alpha}\in LG'_\alpha$ is the Laguerre expansion then $Hf=\Sigma_{n=0}^{\infty}(-1)^n a_{n\alpha}\, l_{n\alpha}$. The relation

$$-4\tau^{-\alpha/2}\left[\tau^{\alpha/2}H_\alpha\left[u^{-\alpha/2}(u^{\alpha/2}f)'\right](\tau)\right]'=(H_\alpha f)(\tau),\ \tau>0,$$

for the Hankel transform shows that it is inconvenient for the operational calculus.

Another transform connected with the Bessel functions and the Bessel equations is so-called B-transform. This transform is introduced in Vladimirov[11] and studied on the spaces of tempered distributions with Laguerre expansions in Pilipović and

Stojanović. The authors[9] in an earlier communication gave the definition, basic properties and the calculus for Bessel transforms b_0 and B_0-transform on the spaces LG_0 and $LG_0' = S_+'$ with the help of Laguerre polynomials. Recall,

$$b_0[\phi](\tau) = -\int_0^\infty \phi'(u) J_0(\sqrt{u\tau})\, du$$

$$= \phi(0) + (1/2)\int_0^\infty \sqrt{\tau/u}\, J_0'(\sqrt{\tau u})\, \phi(u)\, du,\ t > 0,\ \phi \in S_+$$

where J_0 is the Bessel function of the first kind and zero order.

In this paper we give the generalization of the results from Pilipović and Stojanović[9] to the spaces LG_α and LG_α', $\alpha > -1$. We introduce $b_\alpha[\phi](\tau) = \tau^{\alpha/2} b_0 [u^{-\alpha/2}\phi](\tau), \tau > 0,\ \phi \in LG_\alpha$ and B_α as the corresponding transpose mapping $LG_\alpha' \to LG_\alpha'$.

The good operational properties of B_α-transform are the motivation for our investigation. The aim of this approach is solving partial differential equations of the form $P_1\left(\frac{\partial^2}{\partial x^2}x + \frac{\alpha}{x}\right)P_2\left(\frac{\partial}{\partial y}\right)u(x, y) = f(x, y)$ where P_1 and P_2 are polynomials with constant coefficients, and $[0, \infty) \ni y \to f(\cdot, y) \in LG_\alpha'$ is k-times differentiable mapping (see also Remark in section 4). We apply B_α-transform on the vector valued function and by using its Laguerre expansion we obtain the solution in a series form.

In section 2 we recall the definition and the basic properties of the spaces LG_α'. In section 3 we define b_α and B_α-transform through b_0 and B_0 and give their inverses using Laguerre expansions of basic spaces LG_α', $\alpha > -1$. We also give the operational formula for B_α-transform. Section 4 gives application of B_α-transform in solving partial differential equations of the quoted form. We present Laguerre series solutions of them.

2. Basic Spaces

The intrinsic description of the spaces of generalized functions LG_α', $\alpha > -1$ which elements have unique Laguerre orthonormal expansions were given in Duran[4], Pilipović[8], Zayed[12] and Zemanian[13]

Denote by $\{(l_{n,\alpha})_n, n \in \mathbb{N}_0\}$ the Laguerre orthonormal base of the space $L^2((0, \infty))$, where $l_{n,\alpha}(u) = \tau_{n,\alpha} u^{\alpha/2} e^{-u/2} L_n^\alpha(u)$, $u \in (0, \infty), \tau_{n,\alpha} = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2}$ and $L_n^\alpha(u) = \Sigma_{m=0}^n \binom{n+\alpha}{n-m}\frac{(-u)^m}{m!}$, $u \in [0, \infty)$, $n \in \mathbb{N}_0$ are generalized Laguerre polynomials.

By LG_α is denoted the space of smooth functions ϕ on $(0, \infty)$ such that $\phi = x^{\alpha/2}\varphi$, $\varphi \in LG_0$ for which all the seminorms

$$\| \phi \|_{k,n} = \sup_{t \in [0,\infty)} \left\{ t^k \left| \tau^{-\alpha/2} \phi(\tau) \right|^{(n)} \right\}, \; k, n \in \mathbf{N}_0 \qquad \dots (1)$$

are finite. Recall, LG_0 is the space of restrictions to $[0, \infty)$ of rapidly decreasing functions. LG'_0 is the space of tempered distributions supported by $[0, \infty)$ and

$$LG'_\alpha == \{x^{-\alpha/2} f, \; f \in LG'_0\}.$$

On the other hand LG_α is the space of functions

$$\phi \in L^2([0,\infty)) \bigcap C^\infty([0,\infty)), \; \phi = \sum_{n=0}^{\infty} a_{n,\alpha}\, l_{n,\alpha}$$

for which

$$|||\phi|||_s = \left(|a_{0,\alpha}|^2 + \sum_{n=1}^{\infty} |a_{n,\alpha}|^2 n^{2s} \right)^{1/2} < \infty, \; s > 0.$$

Define the derivative $\mathbf{D}_\alpha$ of an $f \in LG'_\alpha$ as follows

$$(\mathbf{D}_\alpha f) = x^{-\alpha/2}\, \partial x^{\alpha/2} f = \frac{\alpha}{2x} f + f' \text{ and } \mathbf{D}_\alpha^{k+1} = \mathbf{D}_\alpha (\mathbf{D}_\alpha^k). \qquad \dots (2)$$

Proposition 1 — (1) The families of seminorms $\{\| \cdot \|_{k,n}, \; k, n \in \mathbf{N}_0\}$, and $\{||| \cdot |||_s, \; s \in \mathbf{N}_0\}$ on LG_α are equivalent.

(2) The mappings $f \mapsto xf$, $f \mapsto \mathbf{D}_\alpha f$, $LG'_\alpha \to LG'_\alpha$ are continuous. Particularly, the mapping $f \mapsto x^k f^k$, $LG'_\alpha \to LG'_\alpha$, is continuous.

PROOF : (1) It follows from Pilipović and Stojanović[9] and the definition of LG_α, since LG'_0 is defined by the family of seminorm (1) with $\alpha = 0$ (see Pilipović[8]).

(2) One can simply prove that the mapping $f \mapsto xf$ is continuous because $xf = x^{-\alpha/2}(xfx^{\alpha/2})$. Since the mappings $LG'_\alpha \to LG'_0$, $f \mapsto x^{\alpha/2} f$, the differentiation in LG'_0 and $LG'_0 \to LG'_\alpha$, $f \mapsto x^{-\alpha/2} f$ are continuous it follows that the mapping $f \mapsto \mathbf{D}_\alpha f$, $LG'_\alpha \to LG'_\alpha$, is continuous. The definition of $\mathbf{D}_\alpha$ implies that the mapping

$$f \mapsto xf' = x\,\mathbf{D}_\alpha f - \alpha/2 f, \; LG_\alpha \to LG'_\alpha$$

is continuous, and by induction one can prove that $f \mapsto x^k f^{(k)}$, $LG'_\alpha \to LG'_\alpha$ is continuous, as well. □

3. b_α AND B_α-TRANSFORM

Let b_α-transform on LG_α be

$$b_\alpha [\phi(u)] (\tau) = \tau^{\alpha/2} b_0 [u^{-\alpha/2} \phi(u)] (\tau) = -\tau^{\alpha/2} \langle (u^{-\alpha/2} \phi(u))', J_0 (\sqrt{u\tau}) \rangle, \tau > 0. \quad \text{... (3)}$$

Proposition 2 — (1) Let $\phi = \Sigma_{n=0}^{\infty} a_{n,\alpha} l_{n,\alpha} \in LG_\alpha$. Then

$$b_\alpha [\phi(u)] = -2 \sum_{n=0}^{\infty} \left\{ \sum_{s=0}^{\infty} (s!)^{-1} (-\alpha)_s (-1)^s \times \left[\sum_{i=n+s+1}^{\infty} \left(\sum_{j=0}^{\infty} a_{i+j,\alpha} (j!)^{-1} \alpha_j \right) + \sum_{j=0}^{\infty} a_{n+s+j,\alpha} (j!)^{-1} \alpha_j \right] \right\} l_{n,\alpha}(\tau), \ \tau > 0.$$

(2) The b_α-transform is a topological isomorphism of LG_α onto LG_α. Particularly, the inverse transform for b_α is b_α, i.e.

$$b_\alpha[b_\alpha[\phi]] = \phi, \ \phi \in LG_\alpha.$$

PROOF : (1) By using formula (39) from Erdelyi *et al.*[6] (p.192) and after ordering the terms we obtain for an $\phi = \Sigma_{n=0}^{\infty} a_{n,\alpha} l_{n,\alpha}$ the Laguerre series form

$$b_\alpha [\phi (u)] = -2 \sum_{n=0}^{\infty} \left\{ \sum_{s=0}^{\infty} (s!)^{-1} (-\alpha_n) (-1)^s \left\{ \left[\sum_{i=n+s+1}^{\infty} \left(\sum_{j=0}^{\infty} a_{i+j,\alpha} (j!)^{-1} \alpha_j \right) + \sum_{j=0}^{\infty} a_{n+s+j,\alpha} (j!)^{-1} \alpha_j \right] \right\} l_{n,\alpha},$$

where

$$(-\alpha)_0 = 1, \ (-\alpha)_m = (-\alpha)(-\alpha+1) \ldots (-\alpha+m-1), \ m \in \mathbb{N}.$$

(2) Since (1) holds and $b_0 [b_0 (\phi)] = \phi$ (Pilipović and Stojanović[9]), it follows

$$b_\alpha [b_\alpha (\phi(u))] (\tau) = u^{\alpha/2} b_\alpha [\tau^{\alpha/2} \tau^{-\alpha/2} b_0 [u^{-\alpha/2} \phi(u)] (\tau)] (u)$$
$$= \tau^{\alpha/2} b_0 [b_0 [u^{-\alpha/2} \phi(u)]] = u^{\alpha/2} u^{-\alpha/2} \phi(u) = \phi(u). \quad \square$$

Define the B_α-transform for an $f \in LG'_\alpha$ as

$$\langle B_\alpha [f], \phi \rangle = \langle f, b_\alpha [\phi] \rangle, \ \phi \in LG_\alpha.$$

Proposition 3 — (1) $B_\alpha [f(u)] (\tau) = \tau^{-\alpha/2} B_0 [u^{\alpha/2} f(u)] (\tau)$, $\tau > 0$.

(2) Let $f = \Sigma_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha} \in LG'_\alpha$, then

$$B_\alpha [f] (\tau) = 2 \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} b_{l,\alpha} \left[\left(\sum_{k=0}^{n-l} (k!)^{-1} (-\alpha)_k \right) \left(\sum_{s=0}^{n} (-1)^s \right.\right.\right.$$

$$\left.\left.\left. \times \left[(1/2) \sum_{j=0}^{n-s} (j!)^{-1} \alpha_j + \sum_{k=0}^{n-1-s} (k!)^{-1} (\alpha+1)_k \right] \right) \right] \right\} l_{n,\alpha}(\tau), \ \tau > 0.$$

(3) B_α-transform is a topological isomorphism of LG'_α onto LG'_α. Particularly, the inverse transform for B_α is B_α, i.e. $B_\alpha[B_\alpha[f]] = f$, $f \in LG'_\alpha$.

(4) Let $f \in LG'_\alpha$, then $B_\alpha \left[\tau f'' + 2 \left(f' + \dfrac{\alpha}{2\tau} f \right) \right] = -(1/4)\, u B_\alpha [f]$.

PROOF : (1) We have

$$\langle B_\alpha [f] (\tau), \phi(\tau) \rangle = \langle f(u), b_\alpha [\phi(\tau)] (u) \rangle = \langle f(u)\, u^{\alpha/2}, b_0 [\tau^{-\alpha/2} \phi(\tau)] (u) \rangle$$

$$= \langle \tau^{-\alpha/2} B_0 [u^{\alpha/2} f(u)] (\tau), \phi(\tau) \rangle$$

which gives 1.

(2) Proposition 2.1 and $\langle B_\alpha [f], l_{n,\alpha} \rangle = \langle f, b_\alpha [l_{n,\alpha}] \rangle$, $n \in \mathbf{N}_0$, imply 2, because of

$$b_\alpha [l_{n,\alpha}(u)] (\tau) = 2 \sum_{q=0}^{n} \left\{ \left(\sum_{k=0}^{n-1} (k!)^{-1} (-\alpha)_k \right) \left(\sum_{s=0}^{n} (-1)^s \right.\right.$$

$$\left.\left. \times \left[(1/2) \sum_{j=0}^{n-s} (j!)^{-1} \alpha_j + \sum_{k=0}^{n-1-s} (k!)^{-1} (\alpha+1)_k \right] \right) \right\} l_{q,\alpha}(\tau), \ \tau > 0.$$

(3) The proof follows from Proposition 2 and

$$B_\alpha [B_\alpha[f]] = u^{-\alpha/2} B_0 [\tau^{\alpha/2} B_\alpha [f]] (u)$$

$$= u^{-\alpha/2} B_0 [B_0 [u^{\alpha/2} f]] = u^{-\alpha/2} u^{\alpha/2} f = f,$$

because $B_0 [B_0[f]] = f$ (Pilipović and Stojanović[9]).

(4) We have

$$b_\alpha [u^{(\alpha/2)+1} (u^{-\alpha/2} \phi)''] (\tau) = \tau^{\alpha/2} b_0 [u^{-\alpha/2} u^{(\alpha/2)+1} (u^{-\alpha/2} \phi)''] (\tau)$$

$$= \tau^{\alpha/2} b_0 [u (u^{-\alpha/2} \phi)''] (\tau) = \tau^{\alpha/2} (-\tau/4)\, b_0 [u^{-\alpha/2} \phi] (\tau)$$

$$= -(\tau/4)\, b_\alpha [\phi] (\tau), \ \tau > 0,$$

because of

$$b_0 [u \phi''] (\tau) = -(\tau/4)\, b_0 [\phi] (\tau),$$ (see Pilipović and Stojanović[9]).

This implies

$$\langle B_\alpha [f(\tau)](u),\ u^{(\alpha/2)+1}(u^{-\alpha/2}\phi(u))''\rangle = -1/4\ \langle(\tau f(\tau)), b_\alpha[\phi(u)](\tau)\rangle$$
$$= -1/4\ \langle B_\alpha[\tau\,\phi(\tau)](u), \phi(u)\rangle.$$

On the other hand

$$\langle f, \tau^{\alpha/2} b_0 [u(u^{-\alpha/2}\phi(u))''](\tau)\rangle = \langle B_0[\tau^{\alpha/2} f(\tau)](u), u(u^{-\alpha/2}\phi)''\rangle$$
$$= \langle\ [uB_0[\tau^{\alpha/2} f(\tau)\,](u)]'', u^{-\alpha/2}\ \phi(u)\rangle$$
$$= \langle 2B_0[\tau^{\alpha/2} f(\tau)(u)] + uB_0''[\tau^{\alpha/2} f(\tau)](u), u^{-\alpha/2}\phi(u)\rangle. \dots \quad (4)$$

We put f'' instead of f in (3) then applying relation (4)

$$B_0[f_\beta * f] = B_0[f^{(\beta)}] = 4^\beta B_0[f]^{-\beta}, \text{ for } \beta \le 0.$$

From Pilipović and Stojanović[9] to B_0-transform we obtain

$$= \langle -2(1/4)[B_0^{-1}(\tau^{\alpha/2} f(\tau)')]'(u) - u(1/16)B_0^{-2}[\tau^{\alpha/2} f(\tau)]''(u), u^{-\alpha/2}\phi(u)\rangle$$
$$= \langle -(1/2)B_0[(\tau f(\tau))'](u) - (1/16)uB_0[\tau^{\alpha/2} f(\tau)](u), u^{-\alpha/2}\phi(u)\rangle$$
$$= \langle -(1/2)B_0\left[\tau^{\alpha/2}\left(f' + \frac{\alpha}{(2\tau)}\right)\right](u) - (1/16)uB_0[\tau^{\alpha/2} f(\tau)], \phi(u)\rangle.$$

Comparing these calculations we obtain

$$-2u^{-\alpha/2} B_0\left[\tau^{\alpha/2}\left(f' + \frac{\alpha}{(2\tau)}\right)\right](u)$$
$$-\ 1/4\ u^{-(\alpha/2)+1} B_0[\tau^{\alpha/2} f](u) = B_\alpha[\tau f](u), u\rangle\ 0$$

and the finishing formula is

$$B_\alpha[\tau f''] + 2B_\alpha\left[f' + \frac{\alpha}{(2\tau)}\right] = -(1/4)\,uB_\alpha[f],$$

or in a shortened form $B_\alpha\left[(\tau f)'' + \frac{\alpha}{\tau} f\right] = -(1/4)\,uB_\alpha[f].$ □

Examples — (1) There holds $f_\beta u^{-\alpha/2} \in LG'_\alpha$, where, for $u > 0$

$$f_\beta(u) = \begin{cases} H(u)\,u^{\beta-1}/\Gamma(\beta) & u > 0 \text{ if } \beta > 0, \\ f^{(N)}_{\beta+N}(u) & u > 0 \text{ if } \beta \le 0,\ \beta + N > 0,\ N \in \mathbf{N}, \end{cases}$$

and (N) is the distributional derivative. We shall find $B_\alpha[u^{-\alpha/2} f_\beta]$.

$$\begin{aligned}\langle B_\alpha [u^{-\alpha/2} f_\beta](\tau), \phi(\tau)\rangle &= \langle u^{-\alpha/2} f_\beta, b_\alpha[\phi(\tau)](u)\rangle \\ &= \langle u^{-\alpha/2} f_\beta(u), u^{\alpha/2} b_0[\tau^{-\alpha/2}\phi(\tau)](u)\rangle \\ &= \langle f_\beta(u), b_0[\tau^{-\alpha/2}\phi(\tau)](u)\rangle \\ &= \langle \tau^{-\alpha/2} B_0[f_\beta(u)](\tau), \phi(\tau)\rangle.\end{aligned}$$

Because of $B_0[f_\beta] = 4^\beta f_{-\beta}$, $\beta \in \mathbf{R}$ (Pilipović and Stojanović[9])

$$B_\alpha[u^{-\alpha/2} f_\beta(u)](\tau) = \tau^{-\alpha/2}(4^\beta f_{-\beta}), \tau > 0.$$

In particularly we have

$$B_\alpha[u^{-\alpha/2}\delta] = \tau^{-\alpha/2} B_0[\delta] = \tau^{-\alpha/2}\delta.$$

$$B_\alpha[u^{-\alpha/2} H] = \tau^{-\alpha/2} B_0[H] = 4\tau^{-\alpha/2}\delta'$$ (Pilipović and Stojanović[9]).

(2) We have

$$x^{-\alpha/2}\delta = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha+1)\tau_{n,\alpha}} l_{n\alpha}(x),$$

where $\tau_{n,\alpha} = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2}$ and the Fourier-Laguerre coefficients are obtained from $\langle x^{-\alpha/2}\delta(x), l_{n\alpha}(x)\rangle = \tau_{n,\alpha}\langle x^{-\alpha/2}\delta(x), x^{\alpha/2} e^{-x/2} L_n^\alpha(x)\rangle = \tau_{n,\alpha} L_n^\alpha(0)$. Because of $L_n^\alpha(0) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}$ [see Erdelyi *et al.*[6], (14), p. 191} we obtain that the Fourier-Laguerre coefficients are $\frac{1}{\tau_{n\alpha}}\left(\frac{1}{\Gamma(\alpha+1)}\right)$, $n \in \mathbf{N}_0$.

(3) Similarly

$$x^{-\alpha/2}\delta' = \sum_{n=0}^{\infty} \frac{1}{\tau_{n,\alpha}}\left[\frac{1}{2\Gamma(\alpha+1)} + \frac{n}{\Gamma(\alpha+2)}\right] l_{n\alpha}.$$

It follows

$$\begin{aligned}-\tau_{n\alpha}(e^{-x/2} L_n^\alpha(x))'(0) &= \tau_{n,\alpha}(1/2\, L_n^\alpha(0) + L_{n-1}^{\alpha+1}(0)) \\ &= \tau_{n,\alpha}\left[1/2\binom{n+\alpha}{n} + \binom{n+\alpha}{n-1}\right] \\ &= \frac{1}{\tau_{n\alpha}}\left(\frac{1}{2\Gamma(\alpha+1)}\right) + \left(\frac{n}{\Gamma(\alpha+2)}\right), \quad n \in \mathbf{N}_0.\end{aligned}$$

4. Application in Solving Partial Differential Equation

Denote by $\mathcal{L}^k([0,\infty), LG'_\alpha)$ the space of continuous mappings

$$y \mapsto f(., y),\ [0, \infty) \to LG'_\alpha$$

which have continuous k-derivatives, $k \in \mathrm{N}$. If $[a, b] \subset [0, \infty)$ then $Q = \{f(., y), y \in [a, b]\}$ is bounded set in LG'_α. Thus, there exist $k > 0$ and $C > 0$ such that

$$\sum_{n=0}^{\infty} | c_{n,\alpha}(y) |^2 n^{-2k} < C, \text{ for all } y \in [a, b],$$

where $f(., y) = \Sigma_{n=0}^{\infty} c_{n,\alpha}(y)\, l_{n,\alpha}$.

First we consider a homogeneous equation

$$P\left(\frac{\partial^2}{\partial x^2} x + \frac{\alpha}{x}\right) u(x, y) = 0,\ x > 0,\ y > 0 \qquad \text{... (5)}$$

in $\mathcal{L}^k([0, \infty), LG'_\alpha)$, where $P(\xi) = \Sigma_{\gamma=0}^{m} a_\gamma \xi^\gamma$, $\xi \in \mathbf{R}$, $a_\gamma \in C$. By applying B_α-transform on (4) and with the notation $\tilde{u}(\tau, y) = B_\alpha[u(x, y)](\tau) = \Sigma_{n=0}^{\infty} a_{n\alpha}(y)\, l_{n\alpha}(\tau)$, we obtain

$$\sum_{\gamma=0}^{m} a_\gamma (-\tau/4)^m \tilde{u}(\tau, y) = 0. \qquad \text{... (6)}$$

We have the following :

Proposition 4 — (1) The solution of the homogeneous equation $\tau^m \tilde{u}(\tau, y) = 0$ in $\mathcal{L}^k([0, \infty), LG'_\alpha)$ is of the form $\tilde{u} = \Sigma_{i=0}^{m-1} a_i(y) \tau^{-\alpha/2} \delta^{(i)}(\tau)$, where $a_i(y) \in C^k [0, \infty)$, $i = 0, \ldots, m - 1$.

(2) The solution of (5) is of the form $\Sigma_{\gamma=0}^{p} a_\gamma(y) \tau^{-\alpha/2} \delta^{(i)}(\tau)$, where $p = \min \{\gamma \mid a_\gamma \neq 0\}$ where a_γ are coefficients of $P(\xi)$, and $a_i(y) \in C^k[0, \infty)$, $i = 0, \ldots, p$.

In particular, the solution of (4) is $B_\alpha^{-1}[\Sigma_{i=0}^{p} a_i(y) \tau^{-\alpha/2} \delta^{(i)}(\tau)]$.

PROOF : (1) Let $y \in (0, \infty)$ be fixed. By using the fact that $\varphi \to \tau^{\alpha/2} \varphi$ is an isomorphism of LG_α onto LG_0 and by

$$\langle \tau^m \tilde{u}(\tau, y),\ \phi(\tau) \rangle = \langle \tau^{-\alpha/2} \tilde{u}(\tau, y), \tau^m \tau^{\alpha/2} \varphi(\tau) \rangle$$

we obtain that

$$\tau^m \tilde{u} = 0 \text{ in } LG'_\alpha \text{ iff } \tau^m \tilde{v} = 0 \text{ in } LG'_0,$$

where $\tilde{v}(\tau, y) = \tau^{\alpha/2} \tilde{u}$. Since $\tau^m \tilde{v}(\tau, y) = 0$ in LG'_0 if and only if $\tilde{v}(\tau, y) = \Sigma_{i=0}^{m-1} a_i(y)\ \delta^{(i)}(\tau)$, it follows $\tilde{u}(\tau, y) = \Sigma_{i=0}^{m-1} a_i(y) \tau^{-\alpha/2} \delta^{(i)}(\tau)$. Since $\tilde{u}(\tau, y) \to \tilde{u}(\tau, y_0)$ in LG'_α if $y \to y_0$ it follows $a_i(y) \in C[0, \infty)$. By similar argument we obtain $a_i(y) \in C^k[0, \infty)$. □

(2) There holds

$$\tau^\gamma \delta^{(i)}(\tau) = \begin{cases} 0, & \gamma > i \\ \dfrac{(-1)^{\gamma} i!}{(i-\gamma)!}\delta^{(i-\gamma)}(\tau), & i, \gamma \in \mathbf{N}_0. \end{cases}$$

By substituting $\Sigma_{i=0}^{s} c_i(y)\,\tau^{-\alpha/2}\,\delta^{(i)}(\tau)$ in (6), where $s > m$, we obtain

$$\sum_{\gamma=0}^{m} \frac{(-1)^\gamma}{4^\gamma} a_\gamma \sum_{i=0}^{s} c_i(y)\,\tau^{-\alpha/2}\,\tau^\gamma\,\delta^{(i)}(\tau) = \sum_{k=0}^{s} b_k\,\tau^{-\alpha/2}\,\delta^{(k)},$$

where

$$b_k = \sum_{j=0}^{m} \frac{a_j\,c_{j+k}(y)}{4^j}\,\frac{(j+k)!}{j!}, \quad k = 0, \ldots, s, j+k \le s.$$

Thus,

$$b_s = a_0\,c_s(y)\,s!, \quad b_{s-1} = a_0\,c_{s-1}(y)\,(s-1)! + \frac{a_1\,c_s}{4}\,s!,$$

....

$$b_{m-1} = a_0\,c_{m-1}(y)\,(m-1)! + \frac{a_1\,c_m}{4}\,m! + \ldots + \frac{a_m\,c_{2m-1}}{4^m}\,\frac{(2m-1)!}{m!}$$

....

$$b_0 = a_0\,c_0 + a_1\,c_1 + \ldots + a_m\,c_m.$$

If $a_0 \neq 0$, it follows $c_s = c_{s-1} = \ldots = c_0 = 0$.

If $a_0 = 0$ and $a_1 \neq 0$ it follows $c_s = c_{s-1} = \ldots = c_1 = 0$, $c_0 \neq 0$. By induction one can prove

if $a_0 = a_1 = \ldots = a_{p-1} = 0$, $a_p \neq 0$, then

$$c_s = c_{s-1} = \ldots = c_p = 0, \; c_0 = 0, \ldots, c_{p-1} \neq 0.$$

Since $a_m \neq 0$, the assertion is proved. □

We will apply our theory in solving vector valued equation of the form

$$P_1\left(\frac{\partial^2}{\partial x^2}x + \frac{\alpha}{x}\right) P_2\left(\frac{\partial}{\partial y}\right) u(x, y) = f(x, y),$$

$$u_y^{(j)}(x, 0) = g_j(x) \in LG'_\alpha, \; j = 0, \ldots, d-1 \qquad \ldots (7)$$

where P_1 and P_2 are polynomials of one variable, d is the order of P_2, $\alpha > -1$ and $[0, \infty) \ni y \to f(., y)$ is an element of $\mathcal{L}([0, \infty), LG'_\alpha)$.

Remark : The equation

$$P_1\left(\frac{\partial^2}{\partial x^2}+\frac{\alpha}{x^2}\right)P_2\left(\frac{\partial}{\partial y}\right)v(x,y)$$
$$= f(x,y),\ v_y^{(j)}(x,0)=g_j(x)\in LG'_\alpha,\ j=0,\ \dots,\ d-1$$

is transformed to (7) by putting $v = xu$. This equation also may be solved by the method which will be given below.

Applying the B_α-transform on the first variable in (6) we obtain

$$P_1(-\tau/4)P_2\left(\frac{\partial}{\partial y}\right)\tilde{u}(\tau,y)=\tilde{f}(\tau,y),\ \tilde{u}_y^{j}=\tilde{g}_j\in LG'_\alpha,\ j=0,\ \dots,\ d-1.$$

This equation may be solved in three steps : (1) By using the Laguerre expansion for $\tilde{u}$ and $\tilde{f}$ in the variable τ, (2) by solving the corresponding coefficient equations with respect to y and (3) by using the inverse transformation B_α^{-1}.

We will explain our method in the simplest case if $P_1(t)P_2(\xi)=t\xi$ and $f\in \mathcal{L}'([0,\infty),LG'_\alpha)$. Then (6) becomes

$$(xu_y(x,y))_{xx}+\frac{\alpha}{x}u_y(x,y)=f(x,y),\ u(x,0)\in LG'_\alpha. \quad \dots (8)$$

We are aimed to find the solution in $\mathcal{L}^1([0,\infty),LG'_\alpha)$. After applying the B_α-transform with respect to x we have

$$-\tau/4\,\tilde{u}_y(\tau,y)=\tilde{f}(\tau,y),\ \tilde{u}(\tau,0)\in LG'_\alpha.$$

With the notation

$$\tilde{u}(\tau,y)=B_\alpha[u(x,y)](\tau)=\sum_{n=0}^{\infty}a_{n,\alpha}(y)\,l_{n,\alpha}(\tau),$$

where $a_{n\alpha}(y)\in C^1[0,\infty)$

and

$$\tilde{f}(\tau,y)=B_\alpha[f(x,y)](\tau)=\sum_{n=0}^{\infty}c_{n,\alpha}(y)\,l_{n,\alpha}(\tau)\ (c_{n\alpha}(y)\in C[0,\infty)),$$

we have

$$(2n+\alpha+1)\,a'_{n,\alpha}(y)-(n+1+\alpha)\,a'_{n+1,\alpha}(y)$$
$$-na'_{n-1,\alpha}(y)=c_{n,\alpha}(y),\ n\in \mathbf{N}_0,$$

because of

$$-xL_n^\alpha=(n+1)L_{n+1}^\alpha-(2n+\alpha+1)L_n^\alpha+(n+\alpha)L_{n-1}^\alpha,\ n\in\mathbf{N}_0$$

[Erdelyi *et al.*[6], (8), p. 188],

or particularly

$$(2n+\alpha+1)\int_0^y a_{n\alpha}'(t)\,dt-(n+1+\alpha)\int_0^y a_{n+1\alpha}'(t)\,dt$$

$$-\,n\int_0^y a_{n-1\alpha}'(t)\,dt = \int_0^y c_{n\alpha}(t)\,dt, \quad n\in \mathbf{N}_0.$$

This gives

$$(2n+\alpha+1)\,[a_{n\alpha}(y)-a_{n\alpha}(0)]-(n+\alpha+1)\,[a_{n+1\alpha}(y)-a_{n+1\alpha}(0)]$$

$$-\,n[a_{n-1\alpha}(y)-a_{n-1\alpha}(0)]=\tilde{c}_{n\alpha}(y),\; n\in\mathbf{N}_0. \qquad \dots (9)$$

Since the coefficients $a_{n\alpha}(0)$ of $\tilde{u}(\tau,0)$ are given we solve the system

$$(2n+\alpha+1)\,a_{n\alpha}(y)-(n+1+\alpha)\,a_{n+1\alpha}(y)-na_{n-1\alpha}(y)$$

$$=\tilde{c}_{n\alpha}(y)+(2n+1+\alpha)\,a_{n\alpha}(0)\,(n+1)\,a_{n+1\alpha}(0)$$

$$-na_{n-1\alpha}(0),\, n\in\mathbf{N}_0,\, a_{-1\alpha}=0. \qquad \dots (10)$$

Having in mind Theorem 1 we conclude that eqn. (7) with the initial condition $u(x,0)\in LG_\alpha'$ is solvable uniquely up to the homogeneous solution of

$$P_1\left(\frac{\partial^2}{\partial x^2}x+\frac{\alpha}{x}\right)u(x,y) = 0$$

which is determined by Theorem 1.

REFERENCES

1. G. Ahuja, *Indian J. Math.*, **28** (1986), 155-62.
2. J. J. Betancor, *Can. J. Math.* **XLI** (1989), 274-84.
3. B. L. J. Braaksma and A. Schuitman, *SIAM J. Math. Anal.* **7** (1976), 771-98.
4. A. Duran, *J. Math. Anal. Appl.* **15** (1990), 166-80.
5. S. J. L. van Eijandhoven, *On Hankel Invariant Distribution Spaces*, EUT-Report 82-WSK-01 (1982), pp. 1-44.
6. A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill Book Co., Inc., New York, 1953.
7. I. Fenyö, *Studia Sci. Math. Hungarica* **23** (1988), 7-14.
8. S. Pilipović, *Com. Rend. Math.* **5** (1989), 23-27.
9. S. Pilipović and M. Stojanović, *Rend. Cir. Matem. Palermo* **33** (1993), 315-426.
10. S. K. Sinha, *Indian J. Math.* **33** (1991), 75-86.
11. V. S. Vladimirov, *Generalized Functions in Mathematical Physics* (in Russian), Nauka, Moscow, 1976.
12. A. Zayed, *SIAM J. Math. Anal.* **13** (1982), 263-79.
13. A. N. Zemanian, *Generalized Integral Transforms*, Interscience, New York, 1968.