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ON THE RATE OF CONVERGENCE OF ONE LINEAR POSITIVE OPERATOR FOR FUNCTIONS OF p-BOUNDED VARIATION*

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In this paper we give the degree of approximation by a general positive operator for the functions of p-bounded variation on $[0, \infty)$. The results here include many results or partial of the previous works.

1. INTRODUCTION

Recently, much effort has been devoted to the investigation on the convergence of linear positive operator for bounded variation. This idea stems from Bojanic *et al.*[1]. In this paper we investigate the rate of convergence of one kind of linear positive operator for functions of p-bounded variation. One will find that it is obvious that is of some generality.

Let $\{x_{jn}; j = 1, ..., n;\ n \geq 1\}$ be a trigular array of independent random variables such that for each fixed n, $x_{1,n}, ..., x_{n,n}$ are identically distributed with $E(x_{1,n}) = \mu_n(x)$ and finite variance $\operatorname{Var}(x_{1,n}) = \sigma_n^2(x) > 0$, where $x \in I \subset [0, \infty)$ is a real parameter. Define

$$S_n = x_{1,n} + \ldots + x_{n,n}.$$

Let h be a well-defined measurable function on $[0, \infty)$ and let $\{a_n\}$ be a sequence of positive numbers. Define an approximation operator by

$$A_n(h, x) = E\{h(a_n S_n)\} = \int_0^\infty h(a_n u)\, dF_{n,x}(u). \quad \ldots (1)$$

when $E\,|h(a_n S_n)| < \infty$, where $F_{n,x}(u)$ is the distribution function of S_n. Define the class $\mathcal{B}_{\mathfrak{A}}$ to be

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$$\mathcal{B}_{\mathcal{A}} = \left\{ f : \begin{array}{l} f \text{ is } p\text{-bounded variation on any finite interval } [0, a] \\ \text{and } f(x) = O(e^{Ax}), \quad x \to \infty, \ A > 0 \end{array} \right\}.$$

Now we define the functuion of bounded variation of order p. The function f is to the bounded variation of order p on $[a, b]$ (BV_p) if

$$\sup \left(\sum_{i=0}^{n} |f(t_{i+1}) - f(t_i)|^p \right)^{1/p} < \infty, \qquad \text{... (2)}$$

where the sup is taken over all partitions $a = t_0, t_1 < \ldots < t_n = b$. The supprem is called the p-variation of function f on interval $[a, b]$, denoted by $V_p(f, [a, b])$, that is

$$V_p(f; [a, b]) = \sup_{\{t_j\}} \left(\sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^p \right)^{1/p}, \qquad \text{... (3)}$$

where $\{t_j\}$ is any partion of $[a, b]$ $a = t_0 < t_1 < \ldots < t_n = b$. Our main results are as follows.

Theorem 1.1 — Let $\{x_{jn}\}_{j=1}^{n}$, $\{a_n\}$ and $\{s_n\}$ define as before. Let $E|x_{1n} - \mu_n(x)| < \infty$ $(r \geq 2)$, $\sigma_n(x) > 0$, $f \in \mathcal{B}_{\mathcal{A}}$. If there exists a constant M such that $E(e^{2A\, a_n S_n}) \geq M (n \to \infty)$. Then, for operator A_n we have

$$|A_n(f, x) - \frac{1}{2}(f(x+) + f(x-))|$$

$$\leq \frac{4\xi_n(x) + x^2}{x^2} \sum_{k=1}^{n} V_p[g_x, x - x/\sqrt{k}, x + x/\sqrt{k}]$$

$$+ \frac{Q_n(x)}{2\sqrt{n}} |f(x+) - f(x-)| + M \frac{\xi_n^{1/2}(x)}{x}.$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq 1; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leq t < x \end{cases}$$

and $V_p(g_x, [a, b])$ denotes the p-total variation of $g_x(t)$ on $[a, b]$;

$$Q_n(x) = \frac{5cE|x_{1,n} - \mu_n(x)|^3}{2\sigma_n^3(x)} + \frac{|a_n \mu_n(x) - x|}{\sqrt{2n}\, a_n \sigma_n(x)}$$

$$\xi_n(x) = E(a_n S_n - x)^2 = n a_n^2 \sigma_n^2 + (x - n a_n \mu_n)^2.$$

Theorem 1.2 — Let $F(x) \in C[0, \infty)$ and for any $x \in [0, \infty)$, there exist $F'_-(x), F'_+(x)$. Assume that $F'_-(x), F'_+(x) \in \mathcal{B}_{\mathcal{A}}$. Other assumptions are the same as

Theorem 1.1. Then, for operator A_n we have

$$|A_n(F,x)-F(x)| \le \frac{4\xi_n(x)+x^2}{x} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} V_p[g_x, x-x/\sqrt{k}, x+x/\sqrt{k}]$$

$$+ \frac{Q_n(x)}{2\sqrt{n}} |f(x+)-f(x-)| + M\frac{\xi_n^{1/2}(x)}{x}.$$

Remark 1 : If x_{jn}, $j = 1, 2, \ldots, n$ are identically distributed for all n, $\mu_n(x) = x$, $\sigma_n^2(x) = \sigma^2(x) > 0$, and $a_n = 1/n$, then $A_n(h,x)$ reduces to the Feller operator. It is simple to verify that a number of classical operators, such as Bernstein, Szasz, Baskaov, Gamma, and Weierstrass operators, are special cases of the Feller operators. So our results imply the results of various authors[3-9] and partial results of Khan[2].

2. Proofs of Main Results

In order to prove our main results we need the following lemmas.

Lemma 2.1 (Khan[2]) —

$$|A_n(\mathrm{sgn}(t-x),x)| \le \frac{2}{\sqrt{n}} Q_n(x),$$

where $Q_n(x)$ is the same as Theorem 1.1.

Lemma 2.2 — Let $F(x)$ satisfy the condition of Theorem 1.2. Then $F(x)$ is absolute continuous on any finite the interval $[0, a]$ and can be represented as

$$F(x) = \int_0^x f(t)\,dt + F(0),$$

where

$$f(t) = \frac{1}{2}(F'_+(t) - F'_-(t)),$$

and $$f(t+) = F'_+(t), f(t) = F'_+(t), f(t) \in \mathcal{B}_{\mathcal{A}}.$$

Proof of Lemma 2.2 — For any finite interval $[0, a]$, $F'_+(t), F'_+(t) \in \mathcal{B}_{\mathcal{A}}$ imply that $F'_+(t), F'_+(t)$ are bound on $[0, a]$. For any $\delta > 0$, taking subintervals $[t_i, t_{i+1}] \subset [0, a]$ $(1 \le i \le n)$ such that

$$\sum_{i=1}^{n} |t_{i+1} - t_i| < \delta.$$

By Lemma 3 of Sun[9], we have

$$\sum_{i=1}^{n} |F(t_{i+1}) - F(t_i)| = \sum_{i=1}^{n} |F(\xi_i)(t_{i+1}) - t_i| \le M_1 \delta,$$

where $M_1 = \sup_{0 \le x \le a} F'_+(x)$. Hence $F(x)$ is absolute continuous on $[0, a]$, $F'_+(x)$ exists almost everywhere on $[0, a]$, and can be represented as

$$F(x) = \int_0^x f(t)\,dt + F(0).$$

If $F'(x)$ exists at x, we have

$$F'(x) = f(x) = F'_+(x) = F'_-(x) = \frac{1}{2}(F'_+(x) + F'_-(x)).$$

If $F'(x)$ does not exist at x, let

$$f(t) = \frac{1}{2}(F'_+(t) + F'_-(t)).$$

We have $f \in \mathcal{B}_{\mathcal{A}}$. On the other hand,

$$F'_+(x) = \lim_{\Delta x \to 0+} \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)\,dt$$

$$= f(x+) + \lim_{\Delta x \to 0+} \frac{1}{\Delta x} \int_x^{x+\Delta x} (f(t) - f(x+1))\,dt.$$

Since $\Delta x \to 0+$ implies $f(t) \to f(x+)$, we have

$$\lim_{\Delta x \to 0+} \frac{1}{\Delta x} \int_x^{x+\Delta x} (f(t) - f(x+))\,dt = 0.$$

Therefore $F'_+(x) = f(x+)$.

Similarly, we can prove that $F'_-(x) = f(x-)$. This complete the proof of Lemma 2.2.

Proof of Theorem 1.1 — It is clear that

$$A_n(f, x) - \frac{1}{2}(f(x+) + f(x-)) = A_n(g_x, x)$$

$$+ \frac{1}{2}(f(x+) - f(x-))\,A_n(\operatorname{sgn}(t-x), x).$$

From Lemma 2.1, it remains to estimate $A_n(g_x, x)$. To this end, let

$$A_n(g_x, x) = \left(\int_0^\alpha + \int_\alpha^\beta + \int_\beta^\gamma + \int_\gamma^\infty \right) g_x(t)\, d\overline{F}_{n,x}(t)$$

$$:= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 \qquad \dots (4)$$

where $\overline{F}_{n,x}(t) = P(a_n S_n \le t)$, $\alpha = x - x/\sqrt{n}$, $\beta = x + x/\sqrt{n}$, $\gamma = 2x$.

We first estimate Δ_2. Since $g_x(x) = 0$, we have

$$| \Delta_2 | = \left| \int_\alpha^\beta g_x(t)\, d\overline{F}_{n,x}(t) \right|$$

$$\le \int_\alpha^\beta | g_x(t) - g_x(x) |\, d\overline{F}_{n,x}(t)$$

$$= \int_\alpha^\beta (| g_x(t) - g_x(x) |^p)^{1/p}\, d\overline{F}_{n,x}(t)$$

$$\le V_p(g_x; [x - x/\sqrt{n},\ x + x/\sqrt{n}\ ]).$$

Hence

$$| \Delta_2 | \le V_p(g_x; [x - x/\sqrt{n},\ x + x/\sqrt{n}\ ]). \qquad \dots (5)$$

In the sequel, we denote $V_p[g_x; [a, b]) = V_p[a, b]$ for the sake of brevity. Now we estimate Δ_1. Note that $\alpha < x$, using partial integration, we have

$$| \Delta_1 | \le \int_0^\alpha (| g_x(x) - g(t) |^p)^{1/p}\, d\overline{F}_{n,x}(t)$$

$$\le \int_0^\alpha V_p[t, x]\, d\overline{F}_{n,x}(t)$$

$$= V_p[x - x/\sqrt{n}, x]\, \overline{F}_{n,x}(\alpha) + \int_0^\alpha \overline{\overline{F}}_{n,x}(t)\, d_t(-V_p[t, x])$$

where $\overline{\overline{F}}_{n,x}(t)$ is the normalized form of $\overline{F}_{n,x}(t)$. We have

$$\overline{\overline{F}}_{n,x}(t) \le \overline{F}_{n,x}(t) = P(a_n S_n \le t)$$

$$\le P(| a_n S_n - x | \ge | t - x |).$$

From Chebyshev's inequality and

$$P(| a_n S_n - x | \ge | t - x |) \le \frac{E(a_n S_n - x)^2}{(t - x)^2}$$

we have

$$\tilde{\tilde{F}}_{n,x}(t) \le \frac{E(a_n S_n - x)^2}{(t-x)^2}.$$

Let

$$\xi_n(t) = E(a_n S_n - x)^2 = n a_n^2 \sigma_n^2 + (x - n a_n \mu_n)^2.$$

Then

$$\bar{F}_{n,x}(\alpha) \le \frac{n}{x^2} \xi_n(x).$$

Thus

$$|\Delta_1| \le \frac{n}{x^2} \xi_n(x) V_p[x - x/\sqrt{n}, x] + \xi_n(x) \int_0^{\alpha} \frac{1}{(x-t)^2} d_t(-V_p[t,x]).$$

Using partial integration again, we have

$$\int_0^{\alpha} \frac{1}{(x-t)^2} d_t(-V_p[t,x]) = -\frac{1}{(x-t)^2} V_p[t,x] \Big|_{t=0}^{\alpha} + \int_0^{\alpha} \frac{2}{(x-t)^3} V_p[t,x]\, dt$$

$$= \frac{1}{x^2} V_p[0,x] - \frac{n}{x^2} V_p[x - x/\sqrt{n}, x] + \int_0^{\alpha} \frac{2}{(x-t)^3} V_p[t,x]\, dt.$$

Put $t = x - x/\sqrt{u}$, then

$$\int_0^{\alpha} \frac{2}{(x-t)^3} V_p[t,x]\, dt$$

$$= \frac{1}{x^2} \int_0^{n} V_p[x - x/\sqrt{u}, x]\, du$$

$$\le \frac{1}{x^2} \sum_{k=1}^{n} V_p[x - x/\sqrt{k}, x].$$

Therefore

$$|\Delta_1| \le \frac{n\xi_n(x)}{x^2} V_p[x - x/\sqrt{n}, x]$$

$$+ \xi_n(x) \left\{ \frac{1}{x^2} V_p[0,x] - \frac{n}{x^2} V_p[x - x/\sqrt{n}, x] + \frac{1}{x^2} \sum_{k=1}^{n} V_p[x - x/\sqrt{k}, x] \right\}$$

$$= \frac{\xi_n(x)}{x^2}\left(V_p[0,x] + \sum_{k=1}^{n} V_p[x - x/\sqrt{k}, x] \right)$$

$$\le \frac{2\xi_n(x)}{x^2} \sum_{k=1}^{n} V_p[x - x/\sqrt{k}, x].$$

That is

$$|\Delta_1| \le \frac{2\xi_n(x)}{x^2} \sum_{k=1}^{n} V_p[x - x/\sqrt{k}, x]. \quad \ldots (6)$$

Similarly, we can get the estimate of Δ_3, that is

$$|\Delta_3| \le \frac{2\xi_n(x)}{x^2} \sum_{k=1}^{n} V_p[x, x + x/\sqrt{k}]. \quad \ldots (7)$$

Now we estimate Δ_4. Since $f(t) = O(e^{At})$, there is a constant $M > 0$, such that

$$|\Delta_4| \le M \int_{2x}^{\infty} e^{At}\, d\overline{F}_{n,x}(t).$$

From Cauchy-Schwarz inequality and the hypothesis of Theorem 1.1

$$E(e^{2Aa_n S_n}) = O(1)$$

we have

$$\int_{2x}^{\infty} e^{2At}\, d\overline{F}_{n,x}(t) \le \left(\int_{2x}^{\infty} e^{At}\, d\overline{F}_{n,x}(t) \right)^{1/2} \left(\int_{2x}^{\infty} d\overline{F}_{n,x}(t) \right)^{1/2}$$

$$= O(1)\, [P(a_n S_n \ge 2x)]^{1/2}.$$

Since

$$P(a_n S_n) \ge 2x) = P(a_n S_n - x) \ge x) = \le P(\,|\,a_n S_n - x\,|\,) \ge x)$$

$$\le \frac{E\,|\,a_n S_n - x\,|^2}{x^2} = \frac{\xi_n(x)}{x^2}.$$

That is

$$|\Delta_4| \le M \frac{\xi_n^{1/2}(x)}{x^2}. \quad \ldots (8)$$

From (4)-(8) we obtain the proof of Theorem 1.1.

Proof of Theorem 1.2 — By Lemma 2.2, $F(t) = \int_0^x f(u)\,du + F(0)$, and

$$f(x) = g_x(t) + \frac{1}{2}(f(x+) + f(x-)) + \frac{1}{2}(f(x+) - f(x-))\,\mathrm{sgn}\,(t-x).$$

Thus

$$F(t) - F(x) = \int_x^t g_x(u)\,du + \left[\frac{1}{2}(f(x+) + f(x-))\right](t-x)$$
$$+ \frac{1}{2}(f(x+) - f(x-)) \int_x^t \mathrm{sgn}\,(u-x)\,du.$$

Since $A_n((t-u), x) = 0$ and

$$\int_x^t \mathrm{sgn}\,(u-x)\,du = |t-u|$$

we have

$$A_n(F, x) - F(x)$$
$$= A_n\left(\int_x^t g_x(u)\,du, x\right) + \frac{1}{2}(f(x+) - f(x-))\,A_n(\mathrm{sgn}\,(t-x), x).$$

By Cauchy-Schwarz inequality we have

$$A_n(|t-x|, x) \le [A_n((t-x)^2, x)]^{1/2} = \xi_n^{1/2}(x).$$

Let $G_x(t) = \int_x^t g_x(u)\,du$. The first term of the right side of (2.13) is

$$A_n(G_x, x) = \left(\int_0^\alpha + \int_\alpha^\beta + \int_\beta^\gamma + \int_\gamma^\infty\right) G_x(t)\,d\,\overline{F}_{n,x}(t)$$
$$:= J_1 + J_2 + J_3 + J_4$$

where $\overline{F}_{n,x}(t)$, α, β, γ are the same as the proof of Theorem 1.1.
We first estimate J_2. $\forall\ t \in [\alpha, \beta]$.

$$|G_x(t)| = \left|\int_x^t g_x(u)\,du\right| \le \int_x^t (|g_x(u) - g_x(x)|^p)^{1/p}\,du$$
$$\le V_p(g_x; [x - x/\sqrt{n}, x + x/\sqrt{n}])\,(t-x)$$

$$\le \frac{x}{\sqrt{n}} V_p \left(g_x; [x - x/\sqrt{n}, x + x/\sqrt{n}\,]\right).$$

Thus

$$|J_2| \le \frac{x}{\sqrt{n}} V_p \left(g_x; [x - x/\sqrt{n}, x + x/\sqrt{n}\,]\right). \qquad \text{... (9)}$$

Now we estimate J_1. Using partial integration, we have

$$J_1 = \int_0^{\alpha} G_x(t)\, d\overline{F}_{n,x}(t) = G_x(\alpha)\overline{F}_{n,x}(\alpha) - \int_0^{\alpha} \overline{F}_{n,x}(t)\, dG_x(t).$$

On the interval $[0, \alpha]$,

$$|G_x| = \left| \int_x^t g_x(u)\, du \right| \le \int_t^x |g_x(u)|\, du := H_x(t).$$

Hence $$|J_1| \le H_x(\alpha)\overline{F}_{n,x}(\alpha) + \int_0^{\alpha} \overline{F}_{n,x}(t)\, d(-H_x(t))$$

$$\le H_x(x - x/\sqrt{n}) \frac{n\xi_n(x)}{x^2} + \xi_n(x) \int_0^{\alpha} \frac{1}{(x-t)^2}\, d(-H_x(t))$$

$$\le \frac{n\xi_n(x)}{x^2} H_x(0) + 2\xi_n(x) \int_0^{\alpha} H_x(t) \frac{1}{(x-t)^3}\, dt.$$

Put $t = x - x/\sqrt{n}$, then

$$\int_0^{\alpha} H_x(t) \frac{2dt}{(x-t)^3} = \frac{1}{x^2} \int_0^{n} H_x(x - x/\sqrt{u})\, du \le \frac{1}{x^2} \sum_{k=1}^{n} H_x(x - x/\sqrt{k}).$$

Note that $g_x(x) = 0$, we have

$$H_x(0) = \int_0^x (|g_x(t) - g_x(x)|^p)^{1/p}\, dt \le xV_p(g_x, [0, x])$$

$$H_x(x - x/\sqrt{n}) = \int_{x - x/\sqrt{k}}^{x} (|g_x(t) - g_x(x)|^p)^{1/p}\, dt$$

$$\le \frac{x}{\sqrt{k}} V_p(g_x, [x - x/\sqrt{k}, x]).$$

Thus

$$|J_1| \leq \frac{\xi_n(x)}{x} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} V_p\,[g_x, x - x/\sqrt{k}, x]. \qquad \text{... (10)}$$

Similarly, we can get the estimate of J_3, that is

$$|J_3| \leq \frac{\xi_n(x)}{x} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} V_p\,[g_x, x, x + x/\sqrt{k}]. \qquad \text{... (11)}$$

Now we estimate J_4. Since $g_x(t) = O(e^{At})$, there is a constant $M > 0$, such that

$$|J_4| \leq \int_{2x}^{\infty} \int_{x}^{t} |g_x(u)|\, du\, d\overline{F}_{n,x}(t)$$

$$\leq M \int_{2x}^{\infty} \int_{x}^{t} e^{Au}\, du\, \overline{F}_{n,x}(t)$$

$$\leq M \int_{2x}^{\infty} \int_{x}^{t} e^{Au}\, \overline{F}_{n,x}(t).$$

Using similar method of Theorem 1.1, we can get

$$|J_4| \leq M \frac{\xi_n^{1/2}(x)}{x^2}. \qquad \text{... (12)}$$

From (9)-(13) we obtain the proof of Theorem 1.2.

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